



الجمهورية الجزائرية الديمقراطية الشعبية
République Algérienne Démocratique et Populaire
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research
جامعة الشاذلي بن جديد - الطارف
University Chadli Bendjedid – El Tarf
كلية العلوم والتكنولوجيا
Faculty des Sciences and Technology
قسم الرياضيات
Department de Mathematics



End of studies dissertation

In order to obtaining the Master's degree

Domain : Mathematics and computer science

Branch : Mathematics

Speciality: Functional Analysis and stochastic calculus

Theme

**Existence and uniqueness of unbounded
solution for nonlinear differential
equation of fractional order**

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Academic Year : 2024–2025

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Juin 2025

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ACKNOWLEDGMENTS

I extend my sincere praise and gratitude to **Allah** Almighty, who bestowed His grace upon me, granted me strength and patience, and illuminated my path with wisdom and determination to complete this work.

I would also like to express my deep appreciation to **Dr. Asma Bouaziz**, my thesis supervisor, for her continuous support, insightful guidance, and valuable contributions throughout all stages of this work. I sincerely thank her from the bottom of my heart for the time she devoted to the realization and final review of this thesis, as well as for her consistent presence and unwavering respect for our scheduled meetings. Her encouragement and expertise played a fundamental role in the successful completion of this final dissertation.

I would also like to express my sincere gratitude to the esteemed members of the jury. I am particularly honored by **Dr. Amar Chidouh**'s acceptance to preside over the defense committee, and I extend my warmest thanks to **Dr. Halima Mechri** for the time she dedicated to reviewing and evaluating this work. Their valuable remarks and critical insight greatly enriched and deepened this dissertation.

My heartfelt thanks go to **my beloved parents**, whom God has commanded me to honor with kindness and devotion. May He grant them long life and good health. This work is dedicated to them in recognition of their endless support and unconditional love.

To **my siblings** who have always stood by my side, to all those who believed in me and supported me, and to my friends and colleagues who shared this journey and left a beautiful mark on my heart.

To myself, who believed in success: be patient and keep going the road is still long. And to all those who live in my heart but whom these lines cannot contain, please accept this work as a token of my gratitude and appreciation.

Dedications

All praise and thanks be to Allah, first and always
Who lit my path, granted me strength and determination, and allowed me to
complete this work.

The journey was neither short nor easy.
It was filled with setbacks and successes, struggles and hope...

But I did it!

I am proud of my perseverance in pursuing my dreams.
This is a moment I have long awaited a dream come true, a story with its final
chapter written.

I dedicate this graduation, the fruit of years of effort and patience:
To the light that guided my way, the one who endured years of hardship so I could
climb the ladder of success,

To the one who cleared the thorns from my path and paved the road to
knowledge...

To my beloved **father: Bekakra Salah,**

This achievement is a humble tribute of loyalty and gratitude.

To the one who taught me morals before letters, the angel of my life,
The one whose prayers carried me, whose kindness soothed my pain...

To my dearest **mother: Boubeker Zaara,**

Today I offer you this knowledge and this degree a dream you gave up to raise and
educate me.

I am forever grateful that God chose you to be my mother.

To the light of my eyes, the peace of my days, my beloved **sisters: Hassiba,
Iman, Rahma, Rahab,**

Thank you for your love, your laughter, and your unwavering support.

To my dear **brother Walid,**

My anchor in life, my strength when I felt weak

May God keep you as a steady shoulder and lasting support.

Grateful to you all. . .

I would not have made it here without God's grace, and without your love and support.

This achievement bears your fingerprints it is your success as much as mine.

ملخص

نظراً لأهمية مبدأ النقطة الثابتة في العديد من المجالات من خلال نمذجة مختلف الظواهر الفيزيائية والبيولوجية والتقنية، مما أثار اهتماماً كبيراً خاصة في دراسة وجود وتفرد الحلول على فترات غير محدودة.

في هذا العمل، وباستخدام صيغة التغير الثابت وبعض المهارات التحليلية، نستخدم مبدأ انكماش باناخ لدراسة وجود وتفرد معادلة تفاضلية كسرية في فضاء باناخ الموزون. يتم توضيح فعالية الطريقة المستخدمة والنتائج التي تم الحصول عليها من خلال مثال.

الكلمات المفتاحية: المعادلة التفاضلية الكسرية؛ مشتقة كابوتو؛ انكماش باناخ؛ تغير الثوابت؛ الوجود؛ التفرد .

Abstract

Given the importance of the fixed point principle in many fields through the modeling of various physical, biological and technical phenomena, it has attracted great interest, particularly in the study of the existence and uniqueness of solutions on unbounded intervals.

In this work, we use the Banach contraction principle to study the existence and uniqueness of a fractional differential equation of fractional orders in weighted Banach space, using the formula for variation of constants formula and a few analytical skills. The effectiveness of the method used and the results obtained are illustrated by an example.

Key Words: Fractional differential equation; Caputo derivative; Banach contraction; variation of constants; existence; uniqueness.

Résumé

Compte tenu de l'importance du principe du point fixe dans de nombreux domaines à travers la modélisation de divers phénomènes physiques, biologiques et techniques, il a suscité un grand intérêt, en particulier dans l'étude de l'existence et de l'unicité des solutions sur des intervalles non bornés.

Dans ce travail, à travers la formule de variation des constantes et quelques compétences analytiques, nous utilisons le principe de contraction de Banach pour étudier l'existence et l'unicité d'une équation différentielle fractionnaire d'ordre fractionnaire dans un espace de Banach pondéré. L'efficacité de la méthode utilisée et les résultats obtenus sont illustrés par un exemple.

Mots clés : Equation différentielle fractionnaire; dérivée de Caputo; contraction de Banach; variation des constantes; existence; unicité.

Notation

$\mathbb{N}$: Set of positive integer numbers.

$\mathbb{N}^*$: $\mathbb{N} - \{0\}$.

$\mathbb{R}$: Set of real numbers.

$\mathbb{R}^+$: Set of non negative real numbers.

$\mathbb{R}^n$: Vector space of dimension n constructed over the field of real numbers.

$\mathbb{C}$: Set of complex numbers .

$(C[a, b], \mathbb{R})$: Space of continuous function from $[a, b]$ into $\mathbb{R}$.

$L^p([a, b])$: Space of integrable function on $[a, b]$.

$BC[a, b]$: Space of absolutely continuous functions on $[a, b]$.

$\|\cdot\|$: Norm.

$\Gamma(\alpha)$: Euler's gamma function.

$B(\alpha, \beta)$: Beta function.

I^α : Riemann-Liouville fractional integral.

D^α : Riemann-Liouville fractional derivative.

${}^cD^\alpha$: Caputo fractional derivative.

(E, T, m) : Measure space.

1 Introduction

Fractional calculus is a non-traditional branch of mathematics. The goal of fractional calculus is to generalize the traditional definition of a fraction. The integer derivative of a function has a derivative of the non-integer order. The earliest known reference to this concept dates back to **september 30, 1695**, when Gottfried Wilhelm Leibniz, in a famous correspondence with L'Hôpital, raised the question of the derivative of order $\frac{1}{2}$. Though considered paradoxical at the time, this idea gradually opened the door to profound mathematical developments.

During the 18th century, notable contributions came from mathematicians such as **Euler (1730)** and **Lagrange (1772)**, followed in the 19th century by **Laplace (1812)** and **Legendre (1817)**. In 1847, Riemann and Liouville provided the first rigorous definition of fractional integration. A significant breakthrough came later with **the introduction of Caputo's derivative (1967)**, which gained popularity due to its effectiveness in modeling physical phenomena. Other major contributors include **Grünwald (1867)**, **Hardy (1927)**, and **Marcel Riesz (1974)**, who enriched the theoretical foundations and practical uses of this field. (See [3] [5] [11] [12] [13] [15] [16]).

The motivation behind fractional calculus arose from the need to describe complex systems with memory effects and non-local dynamics, which classical calculus could not accurately capture. Fractional calculus has proven its importance in various scientific and applied domains, including :

Physics: for modeling anomalous diffusion in porous media.

Electrical engineering: in designing fractional filters with precise frequency responses.

Biology: for simulating the spread of chronic diseases.

Economics: to analyze markets exhibiting nonlinear and chaotic behavior.

This diversity of applications highlights the theoretical depth and representational power of fractional calculus, positioning it as a promising and expanding field in modern mathematics. (See [15] [4] [13] [11] [19]).

Organization of manuscripts :

This document is divided into three chapters, as follows:

- **The first chapter** : We present the necessary preliminary concepts and some definitions and theorems as well as some special functions with their properties (Euler's Gamma and Beta functions). that we used in this memoir.

- **The second chapter** provides a theoretical basis for the necessary fractional calculus. The basic concepts and main properties of order operators are listed.

- The subject of **the third chapter** is the study of the existence and uniqueness of solutions for the following nonlinear hyperbolic fractional differential equation :

$$\begin{cases} {}^c D^\alpha u(t) = f(t, u(t)), & t \geq 0, \\ u(0) = u_0, & u'(0) = u_1, \end{cases} \quad (1)$$

Where $1 < \alpha < 2$, $u_0, u_1 \in \mathbb{R}$, $\mathbb{R}_+ = [0, +\infty)$, $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, ${}^c D^\alpha$ is the right side Caputo fractional derivative. We present a result for the existence of a solution to a fractional differential equation. We first transform the equation (1) into an ordinary differential equation of first-order, then we use the means of the variation of constants formula, we obtain the equivalent integral equations of (1). After that, we prove the existence and uniqueness by using the Banach fixed point theorem.

Equation (1) $\rightarrow$ ODE $\rightarrow$ integral equation.

CHAPTER 1

Preliminary

In this chapter, we introduce some concepts and definitions that we will use in the following chapters .

1 Some definitions

Definition 1.1 (Norm) [8] *Let E be a real or complex vector space. A norm on E is an application, most often denoted by $||\cdot||$:*

$$||\cdot|| : E \rightarrow \mathbb{R}_+ = [0, +\infty[;$$

having the following three properties :

1. $||x|| \geq 0$ for all $x \in E$ and $||x|| = 0 \iff x = 0$.
2. $||\lambda x|| = |\lambda| ||x||$, $\forall x \in E$, $\forall \lambda \in \mathbb{K}$ (homogeneity).
3. $||x + y|| \leq ||x|| + ||y||$, $\forall x, y \in E$ (triangular inequality).

We then say that $(E, ||\cdot||)$ is a normed vector space .

2. Used spaces

Definition 1.2 (Linear operator) [9] A linear operator (or more simply an operator) is a function between two vector spaces which is linear on its domain of definition. If it satisfies :

$$\forall \alpha, \beta \in \mathbb{K}; \forall x, y \in E : A(\alpha x + \beta y) = \alpha A(x) + \beta A(y).$$

Definition 1.3 (Compact operator) Let $D \subset ET$. An operator $F : D \rightarrow E$ is said to be compact if for every bounded set

$S \subset D$, $F(S)$ is relatively compact. Moreover, F is completely continuous if it is continuous and compact.

Definition 1.4 (Lipschitzian application) Let I be an interval of $\mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $k \in \mathbb{R}_+$. We say that f is lipschitzian with ratio k if we have :

$$|f(x) - f(y)| \leq k|x - y|.$$

The function is said to be contracting if we can choose $k < 1$ i.e. if $k \in]0, 1[$.

Definition 1.5 (Equicontinuous set) [2] Let A be a subset of $C(J, \mathbb{R})$, the set A is equicontinuous i.e. for all $\varepsilon > 0$, there exist $\delta > 0$ such that : $|h_1 - h_2| \leq \delta \Rightarrow \|f(h_1) - f(h_2)\| \leq \varepsilon$ for all $h_1, h_2 \in J$ and all $f \in A$.

2 Used spaces

Definition 2.1 (L^p space) [13] Let (E, T, m) be a measured space, $1 \leq p < +\infty$ and f be defined function from E to $\mathbb{R}$, measurable.

1. We say that $f \in L^p = l_{\mathbb{R}}^p(E, T, m)$ if $\int |f|^p dm < +\infty$, we then set :

$$\|f\|_p = \left(\int |f|^p dm \right)^{\frac{1}{p}}.$$

2. We say that $f \notin l^p = l^p_{\mathbb{R}}(E, T, m)$ if $\int |f|^p dm = +\infty$, and we than set $\|f\|_p = +\infty$.

Definition 2.2 (Spaces of continuos functions) [10] Let X be a metric space and $\mathbb{k} = \{\mathbb{R} \text{ or } \mathbb{C}\}$. Let $C(X, \mathbb{k})$ denote the space of continuous of X with value in $\mathbb{k}$.

$C(X, \mathbb{k}) = \{f : X \longrightarrow \mathbb{k}, \text{ continuous}\}$ and $\|f\|_{\infty} = \sup_{x \in X} |f(x)|$.
Then $(C(X, \mathbb{k}), \|\cdot\|_{\infty})$ is Banach space.

Definition 2.3 (Metric space) [20] Let X be a site. A metric or distance is any function :

$$d : X \times X \longrightarrow \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}.$$

Such that, for all $x, y, z \in X$, the following properties hold :

$$(D_1) \quad d(x, y) = 0 \iff x = y.$$

$$(D_2) \quad d(x, y) = d(y, x) \quad (\text{symmetry}).$$

$$(D_3) \quad d(x, z) \leq d(x, y) + d(y, z) \quad (\text{triangle inequality}).$$

The pair (X, d) is called a metric space .

Definition 2.4 (Complete space) [8] A normed space is said to be complete if every Cauchy sequence is convergent.

Definition 2.5 (Banach space) [17] A Banach space is any complete normed vector space over the field $\mathbb{k} = \mathbb{R}$ Or $\mathbb{C}$.

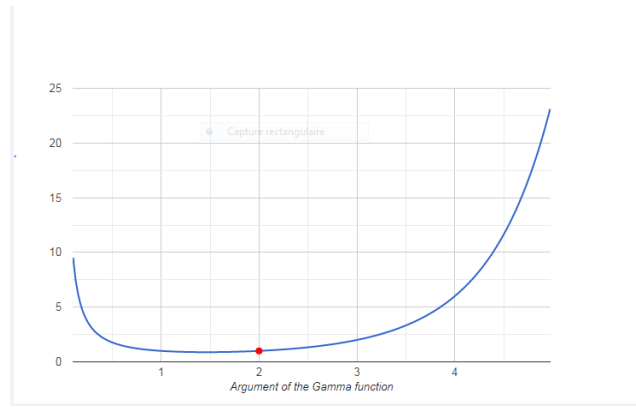
3 Basic tools

3.1 Special functions used in fractional calculus

Gamma function and its property :

Definition 3.1 [7] For $\alpha \in \mathbb{C}$ such that $\text{Re}(\alpha) > 0$. Denoted as $\Gamma(z)$ we define for complex numbers with positive part as :

$$\Gamma(z) : \alpha \longrightarrow \int_0^{+\infty} e^{-t} t^{z-1} dt .$$



Representative curve of Gamma function

Lemma 3.1 (*Convergence and continuity of the Gamma function*) This integral converges for $\text{Re}(z) > 0$ and extends the factorial function beyond integers absolutely on the complex half-plane where the real part is strictly positive. By integrating by parts, we can see that :

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) , \text{Re}(\alpha > 0) .$$

In particular :

$$\Gamma(\alpha + 1) = n! , \forall n \in \mathbb{N} \quad \Gamma(\alpha + 1) .$$

While the integral definition is valid only for $\text{Re}(z) > 0$, the Gamma function

admits an analytic continuation to the complex plane, except for some values where it is not defined.

Proposition 3.1 [4] *The Gamma function has an important property :*

1. *Recurrence relation :*

$$\Gamma(z+1) = z\Gamma(z) .$$

Hence :

$$\Gamma(z) = z! \text{ for } z \in \mathbb{N} .$$

2. *Reflection formula (relating negative and positive values) :*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} .$$

3. *Special values :*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma(1) = 1, \Gamma(0_+) = +\infty.$$

4. *Extension to negative arguments :*

$$\Gamma(x) = \frac{\Gamma(x+n+1)}{x(x+1)\dots(x+n)} \text{ for } -(n+1) < x < -n, n \in \mathbb{N}^* .$$

Example 3.1 *Express the following integral in terms of the Gamma function :*

$$\int_0^\infty x^{\frac{9}{2}} \exp\left(-\frac{1}{2}x\right) dx .$$

3. Basic tools

This is accomplished, by changing variables : $t = \frac{1}{2}x$, as follows :

$$\begin{aligned}\int_0^{\infty} x^{\frac{9}{2}} \exp\left(-\frac{1}{2}x\right) dx &= \int_0^{\infty} (2t)^{\frac{9}{2}} \exp(-t) 2dt \\ &= 2^{\frac{11}{2}} \int_0^{\infty} t^{\frac{11}{2}-1} \exp(-t) dt \\ &= 2^{\frac{11}{2}} \Gamma\left(\frac{11}{2}\right).\end{aligned}$$

Where in the last step we have just used the definition of Gamma function.

Example 3.2 1. $\Gamma(-32) = 43\sqrt{\pi}$.

2. $\Gamma(-12) = -2\sqrt{\pi}$.

3. $\Gamma(32) = 12\sqrt{\pi}$.

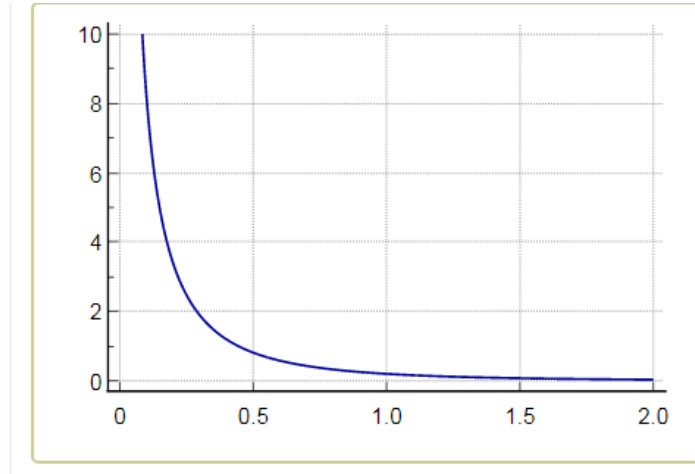
4. $\Gamma(52) = 34\sqrt{\pi}$.

Beta function and its property

Definition 3.2 [15] [14]

The beta function, also known as Euler's integral of the first kind, is a special function that plays an important role in mathematical analysis and has applications in physics and statistics. It is defined as an integral over the interval $[0, 1]$ as follows :

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \text{ for } x, y > 0.$$



Representative curve of Beta function

Proposition 3.2 [4] *The beta function has an important property :*

- *Symmetry property (commutativity) :*

$$B(x, y) = B(y, x) .$$

- *Recursive property :*

$$B(x, y) = \frac{x-1}{x+y-1} B(x-1, y) .$$

- *Integration by parts for the beta function :*

$$B(x+1, y) + B(x, y+1) .$$

- *Special values of the beta function :*

$$B(1, 1) = 1 ; B(1, 2) = \frac{1}{2} ; B(1, 3) = \frac{1}{3} ; B(1, x) = \frac{1}{x} .$$

Example 3.3 For example we have :

$$\begin{aligned} B(2, 3) &= \int_0^1 t(t-1)^2 dt \\ &= \int_0^1 (t - 2t^2 + t^3) dt \\ &= \frac{1}{2}. \end{aligned}$$

Example 3.4 Evaluate $\int_0^1 t^4 (1-t)^3 dt$. The above form can also be written as :

$$\int_0^1 t^{5-1} (1-t)^{4-1} dt.$$

Now, compare the above form with the standard beta function :

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt.$$

So, we get $p = 5$ and $q = 4$.

Using the factorial form of beta function :

$$B(p, q) = \frac{(p-1)! (q-1)!}{(p+q-1)!}.$$

We get :

$$\begin{aligned} B(p, q) &= (4!.3!)/8! \\ &= (4!.6)/8! \\ &= 1/280 . \end{aligned}$$

Therefore, the value of the given expression using beta function is $1/280$.

Relationship between the beta and Gamma functions

Proposition 3.3 *The beta function is closely related to the gamma function through the following formula :*

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

Proof. Prove that :

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

The definition of the Gamma function allows us to write :

$$\Gamma(n) = z^n \int_0^{\infty} e^{-zx} x^{n-1} dx. \quad (1.1)$$

Or :

$$\frac{\Gamma(n)}{z^n} = \int_0^{\infty} e^{-zx} x^{n-1} dx. \quad (1.2)$$

Multiplying both sides by $e^{-z} z^{m-1}$ in equation (1.1) and then integrating with respect to z from $z = 0$ to $z = \infty$.

We get :

$$\Gamma(n) \int_0^{\infty} e^{-z} z^{m-1} dz = \int_0^{\infty} e^{-z} z^{m-1} \left[z^n \int_0^{\infty} e^{-zx} x^{n-1} dx \right] dz.$$

Or

$$\Gamma(n) \Gamma(m) = \int_0^{\infty} \int_0^{\infty} e^{-z(1+x)} z^{m+n-1} x^{n-1} dx dz.$$

4. Banach fixed point theorem

Since the integration limit is same, so changing the order of integration :

$$\begin{aligned}\Gamma(n) \Gamma(m) &= \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(1+x)} z^{m+n+1} dz \right] dx \\ &= \int_0^\infty x^{n-1} \left[\frac{\Gamma(n+m)}{(1+x)^{m+n}} \right] dx .\end{aligned}$$

Or :

$$\begin{aligned}\frac{\Gamma(n) + \Gamma(m)}{\Gamma(n+m)} &= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \\ &= \beta(m, n) .\end{aligned}$$

■

3.2 Comparison between the Gamma and Beta functions

Here we have a comparison between the Gamma and Beta functions :

Feature	Gamma function ($\Gamma(z)$)	Beta function ($B(x, y)$)
Definition	$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$	$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$
Number of variables	single variable z	two variables x and y
Domain	$\mathbb{C}$ except non positive integers	$\mathbb{R}^+$
Relation to factorial	$\Gamma(n) = (n-1)!$	no direct relation to factorial
Interrelation	$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$	$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1)\dots(z+n)}$

4 Banach fixed point theorem

Fixed point theorems are extremely useful tools for solving differential equations. Indeed, these theorems provide sufficient conditions for a given function to admit a

fixed point, thus ensuring the existence and uniqueness of the solution to a given problem by transforming it into a fixed point problem. In this section, we introduce the fixed point theorems that we will use later .

Let $J := [0, T]$, $T > 0$. Let $C(J; \mathbb{R})$ be the Banach space of continuous functions defined from J into $\mathbb{R}$, equipped with the norm :

$$\|x\| = \sup\{|x(t)|, t \in J\}.$$

Definition 4.1 *Let f be a mapping from a set E into itself. A point $u \in E$ is called a fixed point of f if :*

$$f(u) = u.$$

The Banach contraction principal, also known as the Banach fixed point theorem, is a valuable tool in the theory of metric spaces. It guaranteness of fixed points of some applications in metric spaces.

Theorem 4.1 (*Banach fixed point theorem*) *Let E be a complete metric space and let $f : E \rightarrow E$ be a contraction mapping. Then f has unique fixed point.*

Lemma 4.1 *Let E be a Banach space and $A : E \rightarrow E$ an operator. If A is a contraction then A has a unique fixed point .*

CHAPTER 2

Fractional calculus

1 Fractional integral over an interval $[a, b]$

Definition 1.1 [6] Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous function on the interval $[a, b]$. Let us denote by $I_a^1 f$ the interval that vanishes at a . For $t \in [a, b]$:

$$(I_a^1 f)(t) = \int_a^t f(\tau) d\tau.$$

For a second primitive and according to Fubini's theorems we will have :

$$\begin{aligned} (I_a^2 f)(t) &= \int_a^t I_a^1 f(u) du \\ &= \int_a^t \left(\int_a^u f(\tau) d\tau \right) du \end{aligned}$$

$$\begin{aligned}
 &= \int_a^t \left(\int_a^u f(\tau) d\tau \right) f(\tau) d\tau \\
 &= \int_a^t (t - \tau) f(\tau) d\tau.
 \end{aligned}$$

By repeating n times, we obtain the n^{th} primitive of the function f in the form :

$$(I_a^{(n)} f)(t) = \frac{1}{(n-1)!} \int_a^t (t - \tau)^{(n-1)} f(\tau) d\tau, \quad t > 0, \quad n \in \mathbb{N}^*.$$

This formula is called Cauchy's formula, and since the generalization of the factorial by Gamma function :

$$(n-1)! = \Gamma(n).$$

2 Riemann-Liouville fractional integral

Consider a function f such that $f : [a, b] \rightarrow \mathbb{R}$ continuous and integrable. A primitive (antiderivative) of f is given by : $(I_a^1 f)(x) = \int_a^x f(t) dt$. For a double integral we will have :

$$\begin{aligned}
 (I_a^2 f)(x) &= I_a^1 f(I_a^1 f(s))(x) \\
 &= \int_a^x \left(\int_a^s f(t) dt \right) ds \quad (a \leq t \leq s \leq x \leq b) \\
 &= \int_a^x \left(\int_t^x ds \right) f(t) dt \\
 &= \int_a^x (x - t) f(t) dt.
 \end{aligned}$$

2. Riemann-Liouville fractional integral

- For the triple integral we will have :

$$\begin{aligned}
 (I_a^3 f)(x) &= \int_a^x \left(\int_a^s \left(\int_a^t f(\tau) d\tau \right) dt \right) ds \\
 &= \int_a^x \left(\int_a^s (s - \tau) f(\tau) d\tau \right) ds \\
 &= \frac{1}{2} \int_a^x (x - \tau)^2 f(\tau) d\tau.
 \end{aligned}$$

- For the order $n - 1$. We will have :

$$(I_a^n f)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

Definition 2.1 [4] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. The following integral is called the Riemann-Liouville integral of f :

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

Where α is a real (or even complex) number suitably chosen so that $\operatorname{Re}(\alpha) > 0$.

Lemma 2.1 Let $\alpha > 0$. Assume that $u \in C^{n-1}[0, +\infty)$. Then :

$$(I^{\alpha c} D^\alpha u)(t) = u(t) - \sum_{j=0}^{n-1} \frac{u^{(j)}(0)}{j!} t^j. \quad (2.1)$$

Example 2.1 Let the function

$$f(x) = (x-a)^\beta,$$

such that $\beta > -1$, $u \in [a, b]$:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-t)^{\alpha-1} f(t) dt.$$

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-t)^{\alpha-1} (t-a)^\beta dt.$$

By performing the change of variable : $t = a + \tau(x-a)$.

- Where : $\tau = 0$ when $t = a$

-If $\tau = 1$, when $t = x$

- $dt = (x-a)d\tau$.

then it becomes :

$$\begin{aligned} I_a^\alpha (x-a)^\beta &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-a + (x-a)\tau)^{\alpha-1} \left((x-a)^\beta \tau^\beta (x-a) \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a + (x-a)\tau)^{\alpha-1} \left((x-a)^\beta \tau^\beta (x-a) \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^{\alpha-1} (1-\tau)^{\alpha-1} (x-a)^{\beta+1} \tau^\beta d\tau \\ &= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^\beta d\tau \\ &= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha, \beta+1) \\ &= \frac{\Gamma(\alpha) \Gamma(\beta+1)}{\Gamma(\alpha) \Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta} \\ I_a^\alpha (x-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}. \end{aligned}$$

2. Riemann-Liouville fractional integral

Example 2.2 Let the function $f(x) = x^\beta$ with $\beta > 1$ is given by :

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x t^\beta (x-t)^{\alpha-1} dt.$$

By setting $t = xu \Rightarrow dt = xdu$.

If $t = x \Rightarrow u = \frac{x}{x} = 1$ and if $t = 0 \Rightarrow u = 0$.

So the formula becomes :

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (xu)^\beta (x-xu)^{\alpha-1} xdu.$$

Using the beta function, We find :

$$\begin{aligned} I_a^\alpha (x^\beta) &= \frac{1}{\Gamma(\alpha)} \int_0^1 x^\beta u^\beta x^{\alpha-1} (1-u)^{\alpha-1} xdu \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\alpha+\beta-1} u^\beta (1-u)^{\alpha-1} du \\ &= \frac{x^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 u^\beta (1-u)^{\alpha-1} du \\ &= \frac{x^{\alpha+\beta}}{\Gamma(\alpha)} B(\beta+1, \alpha) \\ &= \frac{\Gamma(\beta+1) \Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}. \end{aligned}$$

Thus :

$$I_a^\alpha (x^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}.$$

Theorem 2.1 [9] *For $\alpha > 0$, The Riemann-Liouville integral is linear :*

$$I_a^\alpha(\lambda f(t) + \mu g(t)) = \lambda I_a^\alpha f(t) + \mu I_a^\alpha g(t).$$

Proof.

$$\begin{aligned} & I_a^\alpha(\lambda f(t) + \mu g(t)) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - \tau)^{\alpha-1} (\lambda f + \mu g)(\tau) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x - \tau)^{\alpha-1} \lambda f(\tau) d\tau + \int_a^x (x - \tau)^{\alpha-1} \mu g(\tau) d\tau \right] \\ &= \frac{\lambda}{\Gamma(\alpha)} \int_a^x x(x - \tau)^{\alpha-1} \lambda f(\tau) d\tau + \frac{\mu}{\Gamma(\alpha)} \int_a^x x(x - \tau)^{\alpha-1} \mu g(\tau) d\tau \\ &= \lambda I_a^\alpha f(x) + \mu I_a^\alpha g(x). \end{aligned}$$

■

Theorem 2.2 [15] *Let α and β two complex numbers and $f \in \mathbb{C}^0[a, b]$:*

$$I_a^\alpha(I_a^\beta f) = I_a^{\alpha+\beta} f (Re(\alpha) > 0, Re(\beta) > 0) .$$

$$\frac{d}{dt} I_a^\alpha f(t) = I_a^{\alpha-1} f(t), (Re(\alpha) > 0, Re(\beta) > 0).$$

Proof. 1. For the demonstration we use the beta function. Indeed :

$$\begin{aligned} I_a^\alpha(I_a^\beta f) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - r)^{\alpha-1} (I_a^\beta f(r)) dr \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - r)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^r (r - t)^{\beta-1} f(t) dt \right) dr \end{aligned}$$

2. Riemann-Liouville fractional integral

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-r)^{\alpha-1} dr \int_a^r (r-t)^{\beta-1} f(t) dt.$$

According to Dirichlet's formula :

$$\begin{aligned} I_a^\alpha(I_a^\beta f) &= \frac{B(\alpha, \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha+\beta-1} f(t) dt \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-t)^{\alpha+\beta-1} f(t) dt \\ &= I_a^{\alpha+\beta} f(x). \end{aligned}$$

2. We have :

$$\begin{aligned} \frac{d}{dx}(I_a^\alpha f(x)) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{d}{dx}((x-t)^{\alpha-1} f(t)) dt \\ &= \frac{1}{\Gamma(\alpha)} \frac{d}{dx} \left(\int_a^x (x-t)^{\alpha-1} f(t) dt \right). \end{aligned}$$

Since $f(t)$ and $(x-t)^{\alpha-1}$ are continuous, the application : $t \rightarrow (x-t)^{\alpha-1} f(t)$ is also continuous, and therefore integrable .

■

Proposition 2.1 [15] *Let α be a complex number and $f \in C^0([a, b])$.*

1. $(I_a^0 f)(x) = f(x)$.
2. $\frac{d}{dx}(I_a^\alpha f)(x) = (I_a^{\alpha-1} f)(x)$, $\operatorname{Re}(\alpha) > 1$.
3. $\lim (I_a^\alpha f)(x) = f(x)$, $\operatorname{Re}(\alpha) > 0$.

Proof. 1.

$$\begin{aligned}
 (I_a^0 f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^x f(t) d\left(\frac{(x-t)^\alpha}{-\alpha}\right) \\
 &= \frac{1}{\Gamma(\alpha+1)} \left[f(\alpha)(x-a)^\alpha + \int_a^x f^{(1)}(t)(x-t)^\alpha dt \right].
 \end{aligned}$$

So we have :

$$\begin{aligned}
 (I_a^0 f)(x) &= 1 \left[f(a) \times 1 + \int_a^x f^{(1)}(t) dt \right] \\
 &= f(a) + f(x) - f(a) \\
 &= f(x).
 \end{aligned}$$

2.

$$\begin{aligned}
 \frac{d}{dx} (I_a^\alpha f)(x) &= \frac{d}{dx} \left[\frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \right] \\
 &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{d}{dx} (x-t)^{\alpha-1} f(t) dt \\
 &= \frac{1}{\Gamma(\alpha-1)} \int_a^x (x-t)^{\alpha-2} f(t) dt \quad (\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)) \\
 &= \frac{1}{\Gamma(\alpha-1)} \int_a^x (x-t)^{\alpha-2} f(t) dt \\
 &= (I_a^{\alpha-1} f)(x).
 \end{aligned}$$

2. Riemann-Liouville fractional integral

3. Let $f \in C^0([a, b])$.

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

We can write :

$$(I_a^\alpha 1)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \times 1 dt = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} \rightarrow 1.$$

So :

$$\begin{aligned} |(I_a^\alpha f)(x) - (I_a^\alpha 1)(x) f(x)| &= |(I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x)| \quad (2.2) \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt - \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(x) dt \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |f(t) - f(x)| dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{x-\gamma} (x-t)^{\alpha-1} |f(t) - f(x)| dt + \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{x-\gamma}^x (x-t)^{\alpha-1} |f(t) - f(x)| dt. \end{aligned}$$

As f continues on $[a, b]$, we write :

$$\forall x, t \in [a, b], \forall \varepsilon > 0, \exists \gamma > 0 : |t - x| < \gamma \Rightarrow |f(t) - f(x)| < \varepsilon.$$

Which leads to :

$$\int_{x-\gamma}^x (x-t)^{\alpha-1} |f(t) - f(x)| dt \leq \int_{x-\gamma}^x (x-t)^{\alpha-1} \varepsilon dt = \frac{\gamma^\alpha}{\alpha}. \quad (2.3)$$

On the other hand :

$$\begin{aligned}
 \int_a^{x-\gamma} (x-t)^{\alpha-1} |f(t) - f(x)| dt &\leq \int_a^{x-\gamma} (x-t)^{\alpha-1} (|f(t)| - |f(x)|) dt \\
 &\leq 2 \sup_{\varepsilon \in [a, x]} |f(\varepsilon)| \int_a^{x-\gamma} (x-t)^{\alpha-1} dt, \forall x \in [a, b] \\
 &= 2M \left(\frac{(x-a)^\alpha}{\alpha} - \frac{\gamma^\alpha}{\alpha} \right) \text{ where } M = \sup_{\varepsilon \in [a, x]} |f(\varepsilon)|.
 \end{aligned} \tag{2.4}$$

Comparing between (2.2), (2.3) and (2.4) we find :

$$\begin{aligned}
 |(I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x)| &\leq \frac{1}{\alpha \Gamma(\alpha)} [\varepsilon \gamma^\alpha + 2M((x-a)^\alpha - \gamma^\alpha)] \\
 &= \frac{1}{\Gamma(\alpha+1)} [\varepsilon \gamma^\alpha + 2M((x-a)^\alpha - \gamma^\alpha)].
 \end{aligned}$$

According to the limit :

$$\lim_{\alpha \rightarrow 0^+} |(I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x)| \leq \varepsilon.$$

In other words :

$$\lim_{\alpha \rightarrow 0^+} |(I_a^\alpha f)(x) - \frac{(x-a)^\alpha}{\Gamma(\alpha+1)} f(x)| \leq \varepsilon, \forall \varepsilon > 0.$$

i.e.

$$\lim_{\alpha \rightarrow 0^+} (I_a^\alpha f)(x) = f(x).$$

■

3 Riemann-Liouville fractional derivative

The fractional derivative is a generalization of the classical derivative to non-integer (fractional) orders. It extends the concept of differentiation beyond whole-number derivatives, allowing differentiation of a function to any real or complex order. There are several definitions of fractional derivatives, each with different properties and applications.

Definition 3.1 [15] *Let f be an integrable function on $[a, b]$, then the fractional derivative of order α (with $n - 1 \leq \alpha < n$) in the sense of Riemann-Liouville is defined by :*

$$\begin{aligned}(D_a^\alpha)f(t) &= \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - s)^{n-\alpha-1} f(s) ds \\ &= \frac{d^n}{dt^n} (I^{n-\alpha} f(t)).\end{aligned}$$

Definition 3.2 [22] *The fractional derivative in the Riemann-Liouville sense leads to initial conditions of fractional type that are difficult to explain physically.*

• **Right :**

$$\forall t < \alpha : {}^q_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{-d}{dt}\right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau.$$

f is a function such that ${}_a^r D_t^\alpha$ and ${}_t^R D_b^\alpha$ are defined .

• **Left :**

$$\forall t > \alpha : {}^R_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_\alpha^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau.$$

The fractional derivatives of some functions in the Riemann-Liouville

$f(t)$	$D_t^\alpha f(t) (t > 0, \alpha \in \mathbb{R})$
$\delta(t)$ Dirac	$\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$
t^v	$\frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v+\alpha} (v > -1)$
$e^{-\lambda t}$	$t^{-\alpha} E_{1,1-\alpha}(\lambda t)$
$\cosh(\sqrt{\lambda t})$	$t^{-\alpha} E_{2,1-\alpha}(\lambda t^2)$
$\frac{\sin(\sqrt{\lambda t})}{(\sqrt{\lambda t})}$	$t^{1-\alpha} E_{2,2-\alpha}(\lambda t^2)$
$t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu)$	$t^{\beta-\alpha-1} E_{\mu,\beta-\alpha}(\lambda t^\mu) (\beta > 0, \mu > 0)$

Example 3.1 Let $f(x) = \ln x$. We compute its Riemann Liouville fractional derivative of order α , we have

$$D_{0+}^\alpha f(x) = \frac{d^n}{dx^n} I_{0+}^{n-\alpha} f(x).$$

$$I_{0+}^{n-\alpha} \ln x = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \ln t dt.$$

Using integration by parts and properties of the beta function :

$$I_{0+}^{n-\alpha} \ln x = \frac{x^{n-\alpha} (\ln x - \psi(1-\alpha))}{\Gamma(n-\alpha)}.$$

Where

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z).$$

is the digamma. Differentiating n times :

$$D_{0+}^\alpha \ln x = \frac{x^{-\alpha} (\ln x - \psi(1-\alpha))}{\Gamma(1-\alpha)}.$$

So the final result :

$$D_{0+}^\alpha \ln x = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} (\ln x - \psi(1-\alpha)).$$

3. Riemann-Liouville fractional derivative

Example 3.2 *Let*

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

We have Riemann Liouville fractional derivative formula :

$$D_{0+}^{\alpha} f(x) = \frac{d^n}{dx^n} I_{0+}^{n-\alpha} f(x).$$

$$I_{0+}^{n-\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1+n-\alpha)} x^{p+n-\alpha}.$$

For $p = \frac{1}{2}$:

$$I_{0+}^{n-\alpha} x^p = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+n-\alpha\right)} x^{\frac{3}{2}+n-\alpha}.$$

Using $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$, we get :

$$I_{0+}^{n-\alpha} x^{\frac{1}{2}} = \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{3}{2}+n-\alpha\right)} x^{\frac{3}{2}+n-\alpha}.$$

Using the standard derivative formula for power functions :

$$\frac{d^n}{dx^n} x^q = \frac{\Gamma(q+1)}{\Gamma(q+1-n)} x^{q-n}.$$

Applying this to the previous result :

$$D_{0+}^{\alpha} x^{\frac{1}{2}} = \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma\left(\frac{3}{2}+n-\alpha\right)} \times \frac{\Gamma\left(\frac{3}{2}+n-\alpha\right)}{\Gamma\left(\frac{3}{2}-\alpha\right)} x^{\frac{3}{2}+n-\alpha}.$$

So the final result :

$$D_{0+}^{\alpha} x^{\frac{1}{2}} = \frac{1}{2}\sqrt{\pi} \frac{1}{\Gamma\left(\frac{3}{2}-\alpha\right)} x^{\frac{1}{2}-\alpha}.$$

Proposition 3.1 *We have :*

$${}^{RL}D_a^\alpha \circ {}^{RL}D_a^\beta \neq {}^{RL}D_a^\beta \circ {}^{RL}D_a^\alpha \neq {}^{RL}D_a^{\alpha+\beta}.$$

For all $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$.

Theorem 3.1 [9] *Let $n - 1 < \alpha < n$ and if $f(x)$ satisfies the conditions of Taylor's theorem :*

$$D^\alpha f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k - \alpha + 1)} x^{k-a}.$$

Proof. Since satisfies the conditions of Taylor's theorem, we can apply the Taylor expression :

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} x^k.$$

We differentiate in the Riemann-Liouville sense, we find :

$$D^\alpha f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} D^\alpha x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha}.$$

So :

$$D^\alpha f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} x^{k-a}.$$

■

Lemma 3.1 *Let $\alpha > 0$ and $f \in L^1[a, b]$, then :*

$$D_a^\alpha I_a^\alpha f(x) = f(x).$$

Is true for almost for everywhere $x \in [a, b]$.

4 Caputo fractional derivative

Definition 4.1 [22] *From the fractional derivative with respect to Caputo which includes the initial conditions of the function to be treated as well as its integer derivatives.*

• **Right :**

$$\forall t > \alpha : {}^c D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - \tau)^{m-\alpha-1} f^{(na)}(\tau) d\tau.$$

• **Left :**

$$\forall t < \alpha : b_b^c D_t^B f(t) = \frac{1}{\Gamma(m - \beta)} (-1)^m \int_t^t (\tau - t)^{m-B-1} f^m(\tau) d\tau.$$

Example 4.1 *Let the function : $f(x) = x^\beta$ for $0 < n - 1 < \alpha < n$, we have :*

$${}^C D^\alpha f(x) = I^{n-\alpha} (D^\alpha x^\beta).$$

$$D^n x^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - n)} x^{\beta-n}.$$

$$I_{n-\alpha} \left(\frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - n)} x^{\beta-n} \right) = \frac{\Gamma(\beta + 1)}{\Gamma(n - \alpha) \Gamma(\beta + 1 - n)} \int_0^x (x - t)^{n-\alpha-1} t^{\beta-n} dt.$$

We make the change of variable : $t = yx$ which implies : $dt = xdy$ we then obtain

:

$$\int_0^1 (x - t)^{n-\alpha-1} t^{\beta-n} dt = \int_0^1 (x - xy)^{n-\alpha-1} (xy)^{\beta-n} x dy$$

$$\begin{aligned}
&= \int_0^1 x^{n-\alpha-1} (1-y)^{n-\alpha-1} y^{\beta-n} x^{\beta-n+1} dy \\
&= \int_0^1 x^{\beta-\alpha} (1-y)^{n-\alpha-1} y^{\beta-n} dy \\
&= x^{\beta-\alpha} \int_0^1 (1-y)^{n-\alpha-1} y^{\beta-n} dy \\
&= x^{\beta-\alpha} B(n-\alpha, \beta-n+1) \\
&= x^{\beta-\alpha} \frac{\Gamma(n-\alpha) \Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}.
\end{aligned}$$

Hence :

$$I^{n-\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} x^{\beta-\alpha} \right) = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha) \Gamma(\beta+1-n)} \frac{\Gamma(n-\alpha) \Gamma(\beta-n+1)}{\Gamma(\beta-n+1)} x^{\beta-\alpha}.$$

And finally, we get :

$${}^C D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha) \Gamma(\beta+1-n)} x^{\beta-\alpha}.$$

In particular, for $\beta = 0$, we have :

$${}^C D^\alpha x^0 = {}^C D^\alpha 1 = 0.$$

Unlike the Riemann-Liouville derivation, the fractional-order derivative in the Caputo sense of a constant is zero.

Example 4.2 The Caputo fractional derivative of a constant, $f(x) = C$, is zero.

$${}^C D^\alpha C = 0, \quad \forall C \in \mathbb{R}.$$

$$\begin{aligned} {}^c D_a^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \times 0 dt = 0. \end{aligned}$$

5 The relationship between the Caputo fractional derivative and the fractional derivative with the Riemann-Liouville concept

$$n-1 \geq \alpha < n, n \in \mathbb{N} * -\alpha > 0, f \in \mathbb{C}^n([a, b])$$

$${}^c D_a^\alpha f(t) = {}^R D_a^\alpha f(t) - \sum_{k=0}^{n-1} \frac{(t-k)}{\Gamma(k+1-\alpha)} f^{(k)}(a).$$

Special cases :

1. If $0 < \alpha < 1$ then :

$${}^c D_a^\alpha f(t) = {}^R D_a^\alpha f(t).f(a).$$

2. If $f^{(k)}(a) = 0, n = [\alpha] + 1, \forall k = 0$ so :

$${}^R D_a^\alpha f(t) = {}^c D_a^\alpha f(t).$$

3. If this continues on the ground then :

$${}^c D_a^\alpha I_a^\alpha f(t) = f(t).$$

And from the Caputo fractional differentiation operator is the left opposite of the Riemann-Liouville integrative operator of the same order, but it has no right

5. The relationship between the Caputo fractional derivative and the fractional derivative with the Riemann-Liouville concept

opposite.

Comparison of fractional derivatives (Riemann-Liouville and Caputo)

:

Property	Riemann-Liouville fractional	Caputo fractional
Definition	$\frac{d^n}{dx^n} I_a^{n-\alpha} f(x).$	$I_a^{n-\alpha} \frac{d^n}{dx^n} f(x).$
Formula	$\frac{1}{\Gamma(\alpha)} \int (x-t)^{\alpha-1} f dt.$	$\frac{1}{\Gamma(n-\alpha)} (x-t)^{n-\alpha-1} f^{(n)} dt.$
Initial condition	Fractional order.	Integer order.
Application	Mathematical contexts.	Physical modeling.
Polynomial behavior	$D_a^\alpha x^m \neq 0$ for $m > \alpha$.	$D_a^\alpha x^m = 0$ for $m < \alpha$.

CHAPTER 3

Nonlinear fractional differential equation

In this chapter, we are intressted in the study of uniqueness and existence of unbounded solution for nonlinear differential equations of fractional orders in weighted Banach space. Using Banach contraction principle (see [1]).

1 Position of problem

Let the nonlinear fractional differential equation (FDE_s):

$$\begin{cases} {}^c D^\alpha u(t) = f(t, u(t)). \ t \geq 0, \\ u(0) = u_0, \ u'(0) = u_1, \end{cases} \quad (3.1)$$

Where $1 < \alpha < 2$, $u_0, u_1 \in \mathbb{R}$, $\mathbb{R}_+ = [0, +\infty)$, $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, ${}^c D^\alpha$ is the right side Caputo fractional derivative.

Now, we define the following Banach space which plays a fundamental role in our work .

Let $\rho : \mathbb{R}_+ \rightarrow [1, +\infty)$ be a strictly nondecreasing continuous function with

$$\rho(0) = 1, \ \rho(t) \rightarrow \infty.$$

Let

$$E = \{u \in C[0, +\infty) : \sup_{t \geq 0} |u(t)|/\rho(t) < \infty\}.$$

E is a Banach space endowed with the norm

$$||u|| = \sup_{t \geq 0} \frac{|u(t)|}{\rho(t)}.$$

Furthermore, let :

$$||\varphi||_t = \max_{0 \leq s \leq t} |\varphi(s)|.$$

For any $t \geq 0$, any given $\varphi \in C(\mathbb{R}_+)$.

In the following Lemma, we transform the system (3.1) into an ordinary differential equation of first-order and with a fractional integral term.

Lemma 1.1 *Let $y \in C[0, +\infty)$. Then $u \in C[0, +\infty)$ is a solution of the initial value problem of fractional differential equations*

$$\begin{cases} {}^c D^\alpha u(t) = y(t), & t \in \mathbb{R}_+, 1 < \alpha < 2, \\ u(0) = u_0, & u'(0) = u_1, \end{cases} \quad (3.2)$$

If and only if u is a solution of the initial value problem of first order ordinary differential equations

$$\begin{cases} u'(t) = I_{0+}^{\alpha-1} y(t) + u_1, \\ u(0) = u_0, \end{cases}$$

Proof. We prove that for any $0 < \beta < 1$, if $\psi \in C[0, +\infty)$. Then $(I^\beta \psi)(0) = 0$. In fact, we use the definition of Riemann Liouville fractional integral, since :

$$I^\beta \psi(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \psi(s) ds.$$

Let $v = t - s$ we differentiate $dv = -ds$.

If $s = 0 \rightarrow v = t$ and if $s = t \rightarrow v = 0$.

1. Position of problem

We have : $\psi(t) \leq ||\psi||_t$

$$\begin{aligned} \int_0^t (t-s)^{\beta-1} ds &= \int_t^0 v^{\beta-1} dv \\ &= \left[\frac{v^\beta}{\beta} \right]_0^t \\ &= \frac{t^\beta}{\beta}. \end{aligned}$$

We can conclude that :

$$\begin{aligned} |I^\beta \psi(t)| &= \frac{1}{\Gamma(\beta)} \left| \int_0^t (t-s)^{\beta-1} \psi(s) ds \right| \\ &\leq \frac{||\psi||_t}{\Gamma(\beta+1)} t^\beta \rightarrow 0 \text{ when } t \rightarrow 0 \end{aligned}$$

Let $u \in C[0, +\infty)$ be a solution of the fractional problem (3.2).

For any $t \in \mathbb{R}_+$. The definition of Caputo derivative :

$$\begin{aligned} {}^c D^\alpha u(t) &= I^{n-\alpha} u^{(n)}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds. \end{aligned}$$

Show that :

$${}^c D^\alpha u(t) = ({}^c D^{\alpha-1} D^1 u)(t) = y(t).$$

From (2.1), we have :

$$\begin{aligned}
 I^{\alpha-1} D^{\alpha-1} u'(t) &= I^{\alpha-1} y(t) \\
 u'(t) - \sum_{j=0}^0 \frac{u^{(j)}(0)}{j!} t^j &= I^{\alpha-1} y(t) \\
 u'(t) - u'(0) &= I^{\alpha-1} y(t).
 \end{aligned}$$

As $u'(0) = u_1$ we get :

$$\begin{aligned}
 u'(t) &= u'(0) + I^{\alpha-1} y(t) \\
 &= I^{\alpha-1} y(t) + u_1.
 \end{aligned}$$

Which mean that u is a solution of the problem (3.2).

Conversely, let u be a solution of the problem

$$\begin{aligned}
 {}^c D^\alpha u(t) &= {}^c D^{\alpha-1} u(t) u'(t) \\
 {}^c D^\alpha u(t) &= \underbrace{{}^c D^{\alpha-1} I^{\alpha-1}}_I y + {}^c D^\alpha u_1 \\
 &= I y \\
 {}^c D^\alpha u(t) &= y(t)
 \end{aligned}$$

Also, note that $y \in C[0, +\infty)$. We have $u'(0) = I^{\alpha-1} y(0) + u_1 = u_1$

Finally the problem(3.1) is equivalent to the problem

$$\begin{cases} u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u(0) = u_0. \end{cases} \quad (3.3)$$

■

In the following Lemma, we transform the ordinary differential equation (3.3) into an integral equation.

1. Position of problem

Lemma 1.2 *Let $k \in \mathbb{R}$ satisfies that :*

$$e^{-kt}/\rho(t) \in BC[0, +\infty) \cap L^1[0, +\infty) \quad (3.4)$$

Then (3.3) is equivalent to the following integral equation :

$$u(t) = u_0 e^{-kt} + \frac{1 - e^{-kt}}{k} u_1 + k \int_0^t e^{-k(t-\tau)} u(\tau) d\tau + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha-2} ds f(\tau, u(\tau)) d\tau.$$

Proof. We have :

$$\begin{cases} u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u(0) = u_0 \end{cases}$$

It is clear that we can rewrite (3.3) as follow :

$$\begin{cases} u'(t) = -ku(t) + ku(t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u(0) = u_0 \end{cases} \quad (3.5)$$

Or in other term :

$$\begin{cases} u'(t) + ku(t) = ku(t) + ku(t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u(0) = u_0 \end{cases} \quad (3.6)$$

ODE homogeneous :

$$\begin{cases} u'(t) + ku(t) = 0 \\ u(0) = u_0 \end{cases}$$

We can rewrite :

$$\begin{aligned} \frac{du}{dt} &= -ku \\ \frac{du}{u} &= -k dt \\ \ln u &= -kt + c \\ u &= u_0 e^{-kt} \quad (u(0) = c = u_0) \end{aligned}$$

We use the variation of constant formula :

$$u(t) = u(t)e^{-kt}$$

And we have :

$$u'(t) = (u'(t) - ku(t))e^{-kt}$$

We substitute into the equation (3.6) :

$$\begin{aligned} (u'(t) - ku(t))e^{-kt} + k(u(t)e^{-kt}) &= k(u(t)e^{-kt}) + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u'(t)e^{-kt} - ku(t)e^{-kt} + ke^{-kt}u(t) &= ku(t)e^{-kt} + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \\ u'(t)e^{-kt} &= ku(t)e^{-kt} + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds + u_1 \end{aligned}$$

We use integration to get :

$$\int_0^t u'(t)e^{-kt} dt = \int_0^t [ku(t)e^{-kt} + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-s)^{\alpha-2} f(s, u(s)) ds] dt + u_1$$

We have :

$$\begin{aligned} \int_0^t u'(t)e^{-kt} dt &= u(t)e^{-kt} - u_0e^{-kt} = u(t)e^{-kt} - u_0e^{-kt} \\ &= \int_0^t ku(t)e^{-kt} ds + \int_0^t \frac{1}{\Gamma(\alpha - 1)} \int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) d\tau ds \end{aligned}$$

1. Position of problem

We get :

$$u(t)e^{-kt} - u_0e^{-kt} = k \int_0^t u(s)e^{-ks}ds + \frac{1}{\Gamma(\alpha-1)} \int_0^t \left(\int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) d\tau \right) ds$$

We add u_0e^{-kt} :

$$u(t)e^{-kt} = u_0e^{-kt} + k \int_0^t u(s)e^{-ks}ds + \frac{1}{\Gamma(\alpha-1)} \int_0^t \left(\int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) d\tau \right) ds$$

Finally we get :

$$\begin{aligned} u(t) &= u_0e^{-kt} + e^{-kt} \int_0^t \left[ku(s) + \frac{1}{\Gamma(\alpha-1)} \int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) d\tau + u_1 \right] e^{ks} ds \\ &= u_0e^{-kt} + e^{-kt} \int_0^t ku(s)e^{ks} ds + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^s (s-\tau)^{-k(t-s)} (s-\tau)^{\alpha-2} f(\tau, u(\tau)) ds d\tau + \\ &\quad + u_1e^{-kt} \int_0^t e^{ks} ds \\ &= u_0e^{-kt} + k \int_0^t e^{-k(t-s)} u(s) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^s (s-\tau)^{-k(t-s)} (s-\tau)^{\alpha-2} f(\tau, u(\tau)) ds d\tau + \\ &\quad + u_1e^{-kt} \int_0^t e^{ks} ds \\ &= u_0e^{-kt} + \frac{1-e^{-kt}}{k} u_1 + K \int_0^t e^{-k(t-s)} u(s) ds + \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha-2} ds f(\tau, u(\tau)) d\tau. \end{aligned}$$

On the other hand if :

$$u(t) = u_0 e^{-kt} + \frac{1 - e^{-kt}}{k} u_1 + k \int_0^t e^{-k(t-s)} u(s) ds + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha-2} ds f(\tau, u(\tau)) d\tau,$$

holds we have $u(0) = u_0$. This achieves the proof. ■

2 Existence and uniqueness results

In this section, we will be interested in studying the existence and uniqueness of fixed point of operator A which represent the unique solution of system (3.1).

We define operator $A : E \rightarrow E$ by :

$$Au(t) = u_0 e^{-kt} + \frac{1 - e^{-kt}}{k} u_1 + k \int_0^t e^{-k(t-s)} u(s) ds + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha-2} ds f(\tau, u(\tau)) d\tau. \quad (3.7)$$

For all $t \in \mathbb{R}_+$ and $k \in \mathbb{R}$.

Theorem 2.1 *Assume that hypothesis (3.4) holds and*

(H) *There exists a constant $t > 0$ and a function $\Phi : [0, \infty) \rightarrow [0, \infty)$ so that if $|u| \cdot |v| \leq 1$ then*

(i)

$$|f(t, u) - f(t, v)| \leq \Phi(t) |u - v|. \quad (3.8)$$

For all $t \in \mathbb{R}_+$ and

(ii) *There exists a constant $c \in (0, 1)$ which satisfies*

$$|k| \int_0^\infty \frac{e^{-kt}}{\rho(t)} dt + \sup_{t \geq 0} \int_0^t \frac{k(t-\tau)}{\rho(t-\tau)} \Phi(\tau) d\tau \leq c. \quad (3.9)$$

2. Existence and uniqueness results

Where

$$k(t - \tau) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_{\tau}^t e^{-k(t-s)} (s - \tau)^{\alpha-2} ds, & t \geq \tau \\ 0, & t \leq \tau \end{cases}. \quad (3.10)$$

Then the system (3.1) admit a unique solution in Banach space E .

Proof. We show that $A : E \rightarrow E$ is a contraction mapping. Indeed, let $u, v \in E$, $t \geq 0$ and due to (3.8), we have

$$\begin{aligned} & \left| \frac{Au(t)}{\rho(t)} - \frac{Av(t)}{\rho(t)} \right| \\ = & \left| \frac{u_0 e^{-kt} + \frac{1-e^{-kt}}{k} u_1 + k \int_0^t e^{-k(t-\tau)} u(\tau) d\tau + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_{\tau}^t e^{-k(t-s)} (s - \tau)^{\alpha-2} ds f(\tau, u(\tau)) d\tau}{\rho(t)} - \right. \\ & \left. \frac{v_0 e^{-kt} + \frac{1-e^{-kt}}{k} v_1 + k \int_0^t e^{-k(t-\tau)} v(\tau) d\tau + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_{\tau}^t e^{-k(t-s)} (s - \tau)^{\alpha-2} ds f(\tau, v(\tau)) d\tau}{\rho(t)} \right| \end{aligned}$$

We have : $\frac{u_0 e^{-kt} + \frac{1-e^{-kt}}{k} u_1}{\rho(t)}$ and $\frac{v_0 e^{-kt} + \frac{1-e^{-kt}}{k} v_1}{\rho(t)}$ since they are constants, they vanish to zero, so we have :

$$\begin{aligned} & \left| \frac{Au(t)}{\rho(t)} - \frac{Av(t)}{\rho(t)} \right| \\ = & \left| k \int_0^t \frac{e^{-k(t-\tau)}}{\rho(t)} [u(\tau) - v(\tau)] d\tau + \int_0^t \frac{k(t-\tau)}{\rho(t)} [f(\tau, u(\tau)) - f(\tau, v(\tau))] d\tau \right| \\ \leq & |k| \int_0^t \frac{e^{-k(t-\tau)}}{\rho(t-\tau)} \frac{|u(\tau) - v(\tau)|}{\rho(\tau)} d\tau + \int_0^t \frac{k(t-\tau)}{\rho(t-\tau)} \Phi(\tau) \frac{|u(\tau) - v(\tau)|}{\rho(\tau)} d\tau \\ \leq & |k| \sup_{t \geq 0} \left[\frac{u(\tau) - v(\tau)}{\rho(\tau)} \right] \int_0^t \frac{e^{-k(t-\tau)}}{\rho(t-\tau)} d\tau + \sup_{t \geq 0} \left[\frac{u(\tau) - v(\tau)}{\rho(\tau)} \right] \int_0^t \frac{k(t-\tau)}{\rho(t-\tau)} \Phi(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
 &\leq |k| \|u - y\| \int_0^t \frac{e^{-k(t-\tau)}}{\rho(t-\tau)} d\tau + \|u - v\| \int_0^t \frac{k(t-\tau)}{\rho(t-\tau)} \Phi(\tau) d\tau \\
 &\leq [|k| \int_0^t \frac{e^{-k(t-\tau)}}{\rho(t-\tau)} d\tau + \int_0^t \frac{k(t-\tau)}{\rho(t-\tau)} \Phi(\tau) d\tau] \|u - v\|.
 \end{aligned}$$

Taking into account (3.4) and (3.9). We get :

$$\|Au - Av\| \leq c \|u - v\|.$$

So A is a contraction mapping from E into E . Hence, using the Banach contraction principle and from the last theorem, the operator A has a unique fixed point, which represent the unique solution of system (3.1). ■

Now, we present an illustrative example to support uniqueness and existence results.

Example 2.1 Consider the following initial value problem of Caputo type fractional differential equations

$$\begin{cases} {}^c D^{\frac{3}{2}} u(t) = \omega(t) \left(\frac{1+\alpha u + \sin u}{e^{\varepsilon t}} \right) \\ u(0) = x_0; u'(0) = u_1 \end{cases} \quad (3.11)$$

Where $\omega(t) = e^{\frac{-1}{2}} \frac{1}{e+t^2}$, $e > 0$, $\alpha > -1$, $\lambda > 1$, $k \in \mathbb{R}$, $t \geq 0$. Suppose $0 < |k| \leq \frac{\lambda-1}{2}$ and let $\rho(t) = e^{\lambda t}$, clearly that (3.4) hold and $|k| \int_0^{+\infty} \frac{\lambda-1}{2} \leq \frac{|k|}{\lambda+k}$. Then we define the Banach space by :

$$E_\lambda = \{u(t) \in C[0, +\infty) : \sup_{t \geq 0} \frac{|u(t)|}{e^{\lambda t}} < \infty\}.$$

Equipped with the norm $\|u\| = \sup_{t \geq 0} \frac{|u(t)|}{e^{\lambda t}}$. Clearly

$$|f(t, u) - f(t, v)| \leq \Phi(t) |u - v|.$$

Where $\Phi(t) = (1 + \alpha)\omega(t)$ for all $t \geq 0$.

2. Existence and uniqueness results

$$\begin{aligned}
\frac{k(t-\tau)}{e^{\lambda(t-\tau)}} &= \frac{1}{\Gamma(1/2)} \int_{\tau}^t \frac{1}{e^{(\lambda+k)(t-s)}} \frac{(s-\tau)^{-\frac{1}{2}}}{e^{\lambda(s-\tau)}} ds. \\
&\leq \frac{\int_{\tau}^t \frac{(s-\tau)^{-\frac{1}{2}}}{e^{\lambda(s-\tau)}} ds}{\Gamma(1/2)} \\
&= \frac{\int_0^{t-\tau} \frac{\tau^{\frac{1}{2}}}{e^{\lambda\tau}} d\tau}{\Gamma(1/2)} \\
&\leq \lambda^{1/2}.
\end{aligned}$$

For all $t \geq 0$.

Moreover, if we choose $e \geq (1+\alpha)\lambda^{1/2}\pi(\lambda+k)$ then for all $t \geq 0$, we get :

$$\begin{aligned}
\int_0^t \frac{k(t-\tau)}{e^{\lambda(t-\tau)}} \Phi(\tau) d\tau &= (1+\alpha)\lambda^{1/2}e^{\frac{-1}{2}} \int_0^t \frac{1}{e+\tau^2} d\tau. \\
&= (1+\alpha)\lambda^{1/2}e^{\frac{-3}{2}} \int_0^t \frac{1}{1+(\tau e^{\frac{-1}{2}})^2} d\tau. \\
&= (1+\alpha)\lambda^{1/2}e^{-1} \int_0^{te^{\frac{-1}{2}}} \frac{dz}{1+z^2} \\
&= (1+\alpha)\lambda^{1/2}e^{-1} \arctan(te^{\frac{-1}{2}}) \\
&\leq (1+\alpha)\lambda^{1/2}e^{-1} \frac{\pi}{2} \\
&\leq \frac{1}{2(\lambda+k)} \\
&< 1 - \frac{|k|}{\lambda+k}.
\end{aligned}$$

Then there exists $c = \frac{|k|}{\lambda+k} + \frac{1}{2(\lambda+k)} < 1$ which satisfies

$$||Au - Av|| \leq c||u - v||.$$

So the system(3.11) possesses a unique solution in E_λ follows from the last theorem

.

Conclusion

This work aims to study nonlinear fractional differential equations on an unbounded domain, focusing on proving the existence and uniqueness of solutions. The key to analyzing the problem was integrating the precise properties of fractional derivatives particularly the Caputo derivative with functional analysis tools, such as fixed-point theory, to build a mathematical framework that allows handling the nature of the unbounded domain and the complexity of nonlinear behavior.

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