الجمهورية الجزائرية الديمقراطية الشعبية

People's Democratic Republic of Algeria

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

جامعة الشاذلي بن جديد - الطارف

Chadli Bendjedid-El Tarf University

كلية العلوم و التكنولوجيا

Faculty of Sciences and Technology

قسم الرياضيات..

Department of Mathematics

End of studies dissertation

With a view to obtaining the Master's degree

Domain: Mathematics and Computer Science

Branch: Mathematics

Speciality: Functional Analysis and Stochastic Calculus

Theme

RESEARCH OF UNIQUE COMMON FIXED POINTS IN MULTIPLICATIVE METRIC SPACE

Presented by:

BIACI Rania

Infront of the Jury:

Dr. ALI KHELIL Karima	**MCA**	Badji Mokhtar-Annaba University	President
Dr. BOUHADJERA Hakima	**Prof**	Badji Mokhtar-Annaba University	Supervisor
Dr. LACHOURI Adel	**MCB**	Houari Boumediene Algiers University	Examiner

University Year 2023-2024

Thanks and appreciations

- *Praise is to God, with love, thanks, and gratitude for the beginning and the end.*
- *After years of toil and hardship for the sake of dream and knowledge, I carried within it the wishes of the night, and my trouble today has become a joy to the eye. Here I am standing on the threshold of graduation, reaping the fruits of my toil. O God, praise is to you before you are satisfied, praise is to you if you are satisfied, and praise is to you after you are satisfied.*
- *To those who carry the holiest message in life and those who paved the path of science and knowledge, my honourable teachers, no matter how many tongues speak of your virtues, no matter how many hands describe you, and no matter how much the spirit embodies your meanings, they will still fall short in the face of your efforts, work, and excellence.*
- *In recognition of the assistance provided to bring this work to light, I extend my sincere thanks, appreciation and gratitude to Dr. Bouhadjera Hakima, who supervised this work. She has my sincerest greetings and greatest appreciation for all the guidance she has given me for all your effort and time supervising this study.*
- *I also extend my gratitude, in particular, to Dr. Youbi Zahra who left a beautiful mark on my university life and in my heart with her morals, cooperation, and love. A word of thanks cannot be enough. I ask God to grant her all the success.*
- *Thank you from the bottom of my heart for your constant giving. May God's light guide you and may God reward you with every good thing.*
- *The journey was not short, nor was the road fraught with ease, but I did it. Praise is to God, who made the beginnings easy and we reached the end.*
- *I dedicate this success to my ambitious self first, who started with ambition and ended with success, then to everyone who strived with me to complete my university journey.*
- *To the one who adorned his name with the most beautiful titles, to the one who supported us without limits and gave me without return, to the one who taught me that this world is a struggle and its weapon is knowledge, supporting the first in my journey and my support, strength and refuge after God, my pride (my father).*
- *To the one whom God placed paradise under her feet, whose heart embraced me before her hand and made adversity easy for me with her prayers, to the caring heart and the candle that was for me in the dark nights the secret of strength and success of my paradise (my mother).*
- *To those who supported me with love when I was weak and removed all the troubles from my path, paving the way for me, planting confidence and determination within me, my support and the shoulder on which I always leaned (my brothers).*
- *To all the members of my small family and my large family who supported me and thought well of me and saw the good in me with their eyes and hearts.*
- *And finally, to the companionship of the first step and the penultimate step, to the one who was a rain cloud during the lean years, to the one who believed in me despite my wounds and spread goodness and joy around me, to the only and loyal sister and friend (Taguida Assia).*

Acknowledgments

- *First of all, I would like to address my deep and sincere thanks to Doctor BOUHADJERA, for the interesting subject that she proposed to me.*
  May Doctor BOUHADJERA find here my gratitude for her advice and encouragement, but also for her availability and understanding throughout my research.
- *I also present my sincere thanks to my examiner, Doctor LACHOURI who, by his judicious remarks, allowed me to enrich my work.*
- *Again, I extend my warmest thanks to Doctor ALI KHELIL for agreeing to be president of my dissertation jury.*
- *Finally, I express my deepest gratitude to everyone who contributed directly or indirectly to the achievement of this dissertation.*

Abstract

We are interested in fixed point theory because it is a fully developed branch and is one of the most dynamic fields of research of the last sixty years, with numerous applications in various fields of pure and applied mathematics, as well as in the physical, economic and life sciences. This work consists of three parts and is focused on the study of the existence and uniqueness of common fixed points for mappings which have weak properties. In the first chapter, we presented some famous fixed-point theorems such as Banach fixed-point theorem, Brouwer's one, Schauder theorem and Kakutani's one. In the second chapter, we present the work of Došenović and Radenović [14] after some slight corrections. In the third and last chapter, we improved the results of the second chapter; that is, the Došenović and Radenović's results by removing several conditions and we could proving the existence and uniqueness of common fixed points for four occasionally weakly compatible mappings in a multiplicative metric space under a few conditions.

Keywords and phrases: Multiplicative metric space, occasionally weakly compatible mappings, unique common fixed point theorem.

Résumé

On s’intéresse à la théorie du point fixe car c'est une branche pleinement développée et l’un des domaines de recherche les plus dynamiques des soixante dernières années, avec de nombreuses applications dans divers domaines des mathématiques pures et appliquées, ainsi que dans les domaines physique, économique et sciences de la vie. Ce travail se compose de trois parties et se concentre sur l’étude de l’existence et de l’unicité de points fixes communs pour des applications ayant des propriétés faibles. Dans le premier chapitre, on a présenté quelques théorèmes connus du point fixe tels que le théorème du point fixe de Banach, celui de Brouwer, le théorème de Schauder et celui de Kakutani. Dans le deuxième chapitre, on a présenté le travail de Došenović et Radenović [14] après quelques légères corrections. Dans le troisième et dernier chapitre, on a amélioré les résultats du deuxième chapitre; c'est-à-dire les résultats de Došenović et Radenović en supprimant plusieurs conditions et on a pu prouver l’existence et l’unicité de points fixes communs pour quatre applications occasionnellement faiblement compatibles dans un espace métrique multiplicatif sous quelques conditions.

Mots et expressions clés: Espace métrique multiplicatif, applications occasionnellement faiblement compatibles, théorème du point fixe commun et unique.

ملخص

نحن مهتمون بنظرية النقطة الثابتة لأنها فرع متطور بالكامل وواحدة من أكثر مجالات البحث ديناميكية في الستين عامًا الماضية، مع العديد من التطبيقات في مختلف مجالات الرياضيات البحتة والتطبيقية، وكذلك في الفيزياء والاقتصاد وعلوم الحياة. يتكون هذا العمل من ثلاثة أجزاء ويركز على دراسة وجود وتفرد النقاط الثابتة المشتركة للتطبيقات ذات الخصائص الضعيفة. عرضنا في الفصل الأول بعض نظريات النقطة الثابتة الشهيرة مثل نظرية باناخ للنقطة الثابتة، ونظرية بروير، ونظرية شودر، ونظرية كاكوتاني. وفي الفصل الثاني، نعرض عمل دوسينوفيتش ورادينوفيتش [14] بعد بعض التصحيحات الطفيفة. وفي الفصل الثالث والأخير قمنا بتحسين نتائج الفصل الثاني؛ أي نتائج دوسينوفيتش ورادينوفيتش عن طريق إزالة العديد من الشروط وتمكنا من إثبات وجود وتفرد النقاط الثابتة المشتركة لأربعة تطبيقات متوافقة أحيانًا بشكل ضعيف في مساحة مترية مضاعفة في ظل شروط قليلة.

الكلمات والعبارات المفتاحية: الفضاء المتري المضاعف، التطبيقات المتوافقة بشكل ضعيف في بعض الأحيان، نظرية النقطة الثابتة المشتركة الفريدة.

INTRODUCTION

According to many authors, the prosperous domain of fixed point theory started in the early days of topology with important contributions by Poincare, Lefschetz-Hopf, and Leray-Schauder at the turn of the 19th and early 20th centuries. Fixed point theory is an attractive subject, with a huge number of applications in different fields of mathematics and other branches such as economics, variational inequalities, approximation theory, game theory, and optimization theory, among other areas of interest. It is noticed that, often, fixed points appear when they are needed.

Fixed point is a value that does not change under a given transformation. Specifically, for functions, a fixed point is a point that is mapped to itself by the function. Formally, t is a fixed point of a function f if $f(t) = t$. For example, if f is defined on the real numbers by $f(x) = x^2$ has two fixed points; zero and one because $f(0) = 0$ and $f(1) = 1$. It is noticed that not all functions have fixed points, for instance, $f(x) = x + 3$, has no fixed points, since x is never equal to $x + 3$ for any real number. In graphical terms, a fixed point x means the point $(x, f(x))$ is on the line $y = x$, or in other words the graph of f has a point in common with that line.

On the other hand, Banach's Fixed Point Theorem, also known as The Contraction Theorem, is a significant result in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-mappings of metric spaces, and furnishes a constructive method to find those fixed points. It can be understood as an abstract formulation of Picard's method of successive approximations. According to several authors, the theorem is named after Stefan Banach (1892-1945) who first stated it in 1922. It determined the evolution and growth of the metric fixed point theory.

To improve, extend and generalize Banach's theorem, several authors increase the number of mappings in order to obtain common fixed points. Many others improved the contraction by giving several various conditions. While other mathematicians focused on the complete metric space. For this end, they generalized the last one by introducing many kinds of spaces, in particular, multiplicative metric spaces.

In this dissertation, we will prove the existence and uniqueness of common fixed points for pairs of occasionally weakly compatible mappings in multiplicative metric spaces. Our dissertation is divided into three chapters: in the first chapter, we will give the fundamental fixed point theorems like Banach contraction principle, Brouwer's theorem, Schauder's one and the fixed point-theorem of Kakutani. In the second chapter, we will present the paper of Došenović and Radenović [14] as it is with only some slight corrections. In the third and last chapter, we will improve the Došenović and Radenović's work by removing some conditions, using the concept of occasionally weakly compatible mappings which is more general than some other existing notions. Of course, our dissertation is ended by a conclusion and some cited references.

Contents

Chapter 1

Some Remarkable Fixed Point Theorems

Abstract

*In this chapter, we will present some remarkable fixed point theorems such as Banach (respectively Brouwer, Schauder, and Kakutani) fixed point theorem with some applications. We relied on **Google** to present all these theorems as they exist.*

Keywords: Banach (respectively Brouwer, Schauder, and Kakutani) fixed point theorem.

1.1 Introduction

In mathematics, a fixed-point theorem is a result saying that a function f will have at least one fixed point (a point x for which $f(x) = x$), under some conditions on f that can be stated in general terms. In this chapter, we present the following famous theorems of fixed points.

1.2 Banach's Fixed Point Theorem

Banachs Fixed Point Theorem, also known as The Contraction Theorem, concerns certain mappings (so-called contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point, which we will see is a point that is mapped to itself.

Definition 1.2.1 *Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X if there exists a positive constant $k < 1$ such that*

$$d(Tx, Ty) \leq k.d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

Theorem 1.2.1 *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ (such that $Tx = x$).*

Proof. Let us choose any $x_0 \in X$, and define the sequence (x_n), where

$$x_{n+1} = Tx_n, n = 1, 2, \ldots \tag{1.2}$$

Our proof strategy will be to show that:

1) this sequence is Cauchy;

2) its limit is a fixed point of X;

3) the fixed point is unique.

Step 1: By (1.1) and (1.2), we have that

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1}) \\ &\leq k.d(x_m, x_{m-1}) \\ &= k.d(Tx_{m-1}, Tx_{m-2})) \\ &\leq k^2.d(x_{m-1}, x_{m-2}) \\ &\vdots \\ &\leq k^m.d(x_1, x_0). \end{aligned}$$

Hence by the triangle inequality we get (for $n \geq m$) that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (k^m + k^{m+1} + \cdots + k^{n-1})d(x_1, x_0) \\ &= k^m \frac{1 - k^{n-m}}{1 - k} d(x_0, x_1), \end{aligned}$$

where in the last equality we have used the summation formula for a geometric series. Since $0 < k < 1$, we have $1 - k^{n-m} < 1$, and consequently

$$d(x_m, x_n) \leq \frac{k^m}{1 - k} d(x_1, x_0). \tag{1.3}$$

Since $0 < k < 1$ and $d(x_0, x_1)$ are fixed, it is clear that we can make $d(x_m, x_n)$ as small as we please by choosing m sufficiently large (and $n > m$). This proves that (x_n) is Cauchy. Finally, since (X, d) is complete, there exists an $x \in X$ such that $x_n \to x$.

Step 2: To show that x is a fixed point, we consider the distance $d(x, Tx)$. From the triangle inequality and (1), we get

$$\begin{aligned} d(x, Tx) &\leq d(x, x_m) + d(x_m, Tx) \\ &= d(x, x_m) + d(Tx_{m-1}, Tx) \\ &\leq d(x, x_m) + k.d(x_{m-1}, x) \end{aligned}$$

and since $x_n \to x$ it is clear that we can make this distance as small as we please by choosing m sufficiently large. We conclude that

$$d(x, Tx) = 0 \Rightarrow Tx = x,$$

so $x \in X$ is a fixed point of T.

Step 3: Suppose there are two fixed points $x = Tx$ and $\tilde{x} = T\tilde{x}$. Then from (1.1) it follows that

$$d(x, \tilde{x}) = d(Tx, T\tilde{x}) \leq k.d(x, \tilde{x}),$$

which implies $d(x, \tilde{x}) = 0$ since $0 < k < 1$. Hence $x = \tilde{x}$, and the fixed point x of T is unique.

Note that for **Banach's Fixed Point Theorem** to hold, it is crucial that T is a contraction; it is not sufficient that (1.1) holds for $k = 1$, i.e. that

$$d(Tx, Ty) \leq d(x, y) \quad for\ all\ x, y \in X.$$

To see this, observe that the mappings T_1, $T_2 : \mathbb{R} \to \mathbb{R}$ given by $T_1 x = x + 1$ and $T_2 x = x$ both satisfy (1.1) with $k = 1$. The mapping T_1 has no fixed points, whereas T_2 has infinitely many. ■

Applications

The most interesting applications of Banach's Fixed Point Theorem arise in connection with function spaces. The theorem then yields existence and uniqueness results for differential equations, as we will now see.

Application to Integral Equations

we consider integral equations of the from

$$f(x) = g(x) + \lambda \int_a^b k(x,y)f(y)dy, \tag{1.4}$$

where $f : [a,b] \to \mathbb{R}$ is an unknown function, $k : [a,b] \times [a,b] \to \mathbb{R}$ is a given function (called the **kernel**) and λ is a parameter. Such integral equations can be considered in various function spaces. In this section we consider (1.4) only in $(C[a,b], d_\infty)$. We assume that $g \in C[a,b]$, and that the kernel k is continuous on the square $[a,b] \times [a,b]$.

Theorem 1.2.2 *The metric space of continuous functions $C[a,b]$ with the uniform metric d_∞ is complete.*

Recall that the uniform metric d_∞ is given by

$$d_\infty(f,g) = \|f-g\|_\infty = \sup_{a \le x \le b} |f(x) - g(x)|, \;\; f,g \in C[a,b].$$

Equation (1.4) can be restated as $T(f) = f$, where

$$T(f)(x) = g(x) + \lambda \int_a^b k(x,y)f(y)dy. \tag{1.5}$$

Since g and k are both continuous, this defines an operator $T : C[a,b] \to C[a,b]$. Let us now determine for which values of λ the mapping T is a contraction. Note first that since k is continuous, it must also be bounded

$$k(x,y) \le c \text{ for all } (x,y) \in [a,b] \times [a,b]. \tag{1.6}$$

We have

$$
\begin{aligned}
d_\infty(T(f_1), T(f_2)) &= \max_{a\le x\le b} |T(f_1)(x) - T(f_2)(x)| \\
&= |\lambda| \max_{a\le x\le b} \left| \int_a^b k(x,y)(f_1(y) - f_2(y))dy \right| \\
&\le |\lambda| \max_{a\le x\le b} \int_a^b |k(x,y)||f_1(y) - f_2(y)|dy \\
&\le c|\lambda| \max_{a\le x\le b} |f_1(x) - f_2(x)| \int_a^b dy \\
&= c|\lambda|(b-a)d(f_1, f_2).
\end{aligned}
$$

Recall that T is a contraction if

$$d(T(f_1), T(f_2)) \le kd(f_1, f_2) \text{ for all } f_1, f_2 \in C[a,b]$$

for some constant $0 < k < 1$, and we see that this is indeed the case if

$$|\lambda| < \frac{1}{c(b-a)}. \tag{1.7}$$

In light of Theorem (1.2.2), Banachs Fixed Point Theorem now gives:

Theorem 1.2.3 *Suppose k and g in (1.4) are continuous on $[a,b] \times [a,b]$ and $[a,b]$, respectively, and assume that the parameter λ satisfies (1.7), with c defined in (1.6). Then the integral equation (1.4) has a unique solution $f \in C[a,b]$. This solution is the limit of the iterative sequence $(f_0, f_1, f_2, \ldots)$, where f_0 is any continuous function on $[a,b]$, and*

$$f_{n+1}(x) = g(x) + \lambda \int_a^b k(x,y) f_n(y)dy, \;\; n = 1, 2, \ldots$$

Application to Differential Equations

Let us consider the following initial value problem (say (P))

$$x'(t) = \frac{dx}{dt} = f(t, x),\ x(t_0) = x_0,$$

where $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is a given function and $x(t)$ is an unknown function which we wish to determine. In this subsection we will use Banachs Fixed Point Theorem to prove the famous Picard-Lindelöf Theorem, which guarantees the uniqueness and existence of a solution to (P).

Theorem 1.2.4 *(Picard-Lindelöf) Let be continuous on a rectangle*

$$R = (t, x) : |t - t_0| \le a,\ |x - x_0| \le b,$$

and thus bounded on R, say $|f(x, t)| \le c$. Suppose that f satisfies a Lipschitz condition on R with respect to its second argument, meaning there exists a constant k such that

$$|f(t, x) - f(t, y)| \le k|x - y| \qquad \textit{for all } (t, x), (t, y) \in R.$$

Then the initial value problem (P) has a unique solution which exists on an interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta < min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}. \tag{1.8}$$

Proof. We split the proof into five steps.

Step 1: Equivalent formulation as an integral equation: We observe first that if a function $x \in C^1[t_0 - a, t_0 + a]$ solves (P), then necessarily the following problem (say (Q))

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

by integration. On the other hand, if $x \in C[t_0 - a, t_0 + a]$ fulfils (Q), then x is a continuously differentiable solution to (P) (this follows from the Fundamental Theorem of Calculus). Thus, the initial value problem (P) for $x \in C_1[t_0 - a, t_0 + a]$ is equivalent to (Q) for $x \in C[t_0 - a, t_0 + a]$.

Step 2: Constructing an operator T on a complete space to which we can apply Banach's Fixed Point Theorem: For $J = [t_0 - \beta, t_0 + \beta]$ and $y \in C(J)$, define the operator

$$T(y)(t) := x_0 + \int_{t_0}^{t} f(s, y(s))ds, \; t \in J.$$

Consider the set

$$X := \left\{ y \in C(J) : y(t_0) = x_0, \; \sup_{t \in J} |x_0 - y(t)| \leq c\beta \right\}.$$

This is a closed subspace of $C(J)$ (endowed with the metric d_∞), so (X, d_∞) is complete.

Step 3: observe that $T : X \to X$ for $y \in X$, we need to show that $T(y) \in X$.

Observe that $T(y)(t_0) = x_0$. Moreover, we have

$$|x_0 - T(y)(t)| = \left| \int_{t_0}^{t} f(s, y(s))ds \right| \leq |t - t_0|. \max_{t \in J} |f(t, y(t))| \leq c\beta,$$

so $T(y) \in X$.

Step 4: Showing T is a contraction: Fix $y_1, y_2 \in X$. We have

$$\begin{aligned} |T(y_1)(t) - T(y_2)(t)| &= \left| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s)) ds \right| \\ &\leq |t - t_0| . \max_{s \in J} k |y_1(s) - y_2(s)| \\ &\leq k\beta d(y_1, y_2). \end{aligned}$$

The right hand side above is independent of t, so taking the maximum over $t \in J$ on both sides, we get

$$d(T(y_1), T(y_2)) \leq k\beta d(y_1, y_2).$$

Recalling (1.8), we see that $k\beta < 1$, so T is a contraction on X.

Step 5: Conclusion: Banach's Fixed Point Theorem implies that T has a unique fixed point $x \in X$ such that

$$x(t) = T(x)(t) = x_0 + \int_{t}^{t_0} f(s, x(s)) ds.$$

It thus follows from Step 1 that (P) has a unique, continuous solution $x(t)$ on the interval $[t_0 - \beta, t_0 + \beta]$.

■

1.3 The Brouwer and Schauder Fixed Point Theorem

The Brouwer Fixed-Point Theorem is one of the most important existence principles in mathematics. It has interesting applications to game theory, mathematical economics, and numerical mathematics. Further important existence principles in mathematics are **the Hahn-Banach theorem**, the Weierstrass existence theorem for minima, and the Baire category theorem. **The Schauder Fixed Point Theorem** is an extension of the Brouwer Fixed Point Theorem.

Theorem 1.3.1 *(Brouwer Fixed Point Theorem - Version 1) Any continuous mapping of a closed ball in* $\mathbb{R}^n$ *into itself must have a fixed point.*

Theorem 1.3.2 *(Brouwer Fixed Point Theorem - Version 2) Let* $(X, \|.\|)$ *be a finite-dimensional normed space and* $S \subset X$ *is compact, convex, and nonempty. Any continuous operator* $A : S \to S$ *has at least one fixed point.*

Example 1.3.1 *(Counter Examples) The following counter examples show the essentials of each assumption in the Brouwer Fixed-Point Theorem (version 2).*

- $S = [0, 1]$ compact, convex and nonempty, but $A : S \to S$ not continuous and the graph $y = A(x)$ does not cross the diagonal $y = x$. No fixed point.

- $S = R$ and $A : S \to S$, $A(x) = x + 1$. A is continuous, S is convex, nonempty, but not compact. No fixed point.

- Let S be a closed annulus and $A : S \to S$ is a rotation of the annulus around the center. A proper rotation is fixed-point free. In this case, S is compact, nonempty but not convex.

Theorem 1.3.3 *(Schauder Fixed Point Theorem - Version 1) Let* $(X, ||.||)$ *be a Banach space over* $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$) *and* $S \subset X$ *is closed, bounded, convex, and nonempty. Any compact operator* $A : S \to S$ *has at least one fixed point.*

Theorem 1.3.4 *(Schauder Fixed Point Theorem - Version 2) Let* $(X, ||.||)$ *be a Banach space and* $S \subset X$ *is compact, convex, and nonempty. Any continuous operator* $A : S \to S$ *has at least one fixed point.*

Applications

Applications to Ordinary Differential Equations

Theorem 1.3.5 *(The Peano Theorem) Given* $(x_0, u_0) \in \mathbb{R}^2$*, let* $F(x, w)$ *be a real-valued continuous function on a rectangle*

$$S = (x, w) \in \mathbb{R}^2 : |x - x_0| \leq \textit{ and } |w - u_0| \leq b,$$

denote $c = \max_{(x,w)\in S} |F(x, w)|$*. Then for* $0 < h \leq a$ *and* $hc \leq b$*, the following initial value problem*

$$\begin{cases} u' = F(x, u), \qquad x_0 - h \leq x \leq x_0 + h \\ u(x_0) = u_0. \end{cases} \tag{1.9}$$

has at least one solution.

Proof. Denote $X := C[x_0 - h, x_0 + h]$ and $M := \{u \in X : \|u - u_0\|_\infty \leq b\}$. For each $u \in M$, consider the following operator A

$$Au(x) := u_0 + \int_{x_0}^{x} F(y, u(y))dy, \text{ for } x \in [x_0 - h, x_0 + h].$$

Similar to the part of the Picard-Lindelöf theorem, we have $A : M \to M$. Next, we will prove that A is continuous and $A(M)$ is bounded and equicontinuous. Since $A(M) \subset M$, the set $A(M)$ is bounded. The continuous of A and the equicontinuous of $A(M)$ come from the following inequality:

$$|Au(x) - Au(z)| = \left| \int_{z}^{x} F(y, u(y))dy \right| \leq c|z - x|.$$

By the Arzela-Ascoli Theorem, the set $A(M)$ is relatively compact in X. Since M is bounded, this implies $A : M \to M$ is a compact operator. Moreover, the closed ball

M is closed, bounded, convex, and nonempty. By the Schauder fixed point theorem, the equation

$$Au = u, \ u \in M$$

has a solution $u_* \in M$. Differentiating the integral equation with respect to x, we see that u_* is also a solution of the (1.9). ■

1.4 Kakutani Fixed Point Theorem

In mathematical analysis, **the Kakutani fixed-point theorem** is a fixed-point theorem for set-valued functions. It provides sufficient conditions for a set-valued function defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. **The Kakutani fixed point theorem** is a generalization of **the Brouwer fixed point theorem**. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces. **Kakutani's theorem** extends this to set-valued functions.

Theorem 1.4.1 *Let S be a non-empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $\varphi : S \to 2^S$ be a set-valued function on S with the following properties:*

- *φ has a closed graph.*
- *$\varphi(x)$ is non-empty and convex for all $x \in S$.*

Then φ has a fixed point.

Definition 1.4.1 *(Set-valued function) A set-valued function φ from the set X to the set Y is some rule that associates one or more points in Y with each point in X.*

Formally it can be seen just as an ordinary function from X to the power set of Y, written as $\varphi : X \to 2^Y$, such that $\varphi(x)$ is non empty for every $x \in X$. Some prefer the term correspondence, which is used to refer to a function that for each input may return many outputs. Thus, each element of the domain corresponds to a subset of one or more elements of the range.

Definition 1.4.2 *(Closed graph) A set-valued function $\varphi : X \to 2^Y$ is said to have a closed graph if the set $(x, y)|y \in \varphi(x)$ is a closed subset of $X \times Y$ in the product topology; i.e., for all sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $x_n \to x,\ y_n \to y$ and $y_n \in \varphi(x_n)$ for all n, we have $y \in \varphi(x)$.*

Definition 1.4.3 *(Fixed point) Let $\varphi : X \to 2^X$ be a set-valued function. Then $a \in X$ is a fixed point of φ if $a \in \varphi(a)$.*

Chapter 2

Multiplicative Metric Spaces and Contractions of Rational Type

Abstract

In this part, we present the paper of Došenović and Radenović [14] as it is, with only slight corrections. So, the main purpose of this chapter is to study the fixed point theorems with contractions of rational type in multiplicative metric spaces. We analyzed whether it was possible to get better results in the context of metric spaces.

Keywords: Metric Space, Common Fixed Point, Multiplicative Metric Space, Cauchy sequence.

2.1 Introduction and Preliminaries

In 2008, Bashirov et al., defined new kind of spaces, called multiplicative metric spaces in the following way:

Definition 2.1.1 *([9]) Let* $X \neq \emptyset$. *An operator* $d^* : X \times X \to \mathbb{R}$ *is a multiplicative metric on* X, *if it satisfies:*

$(m1^*)$ $d^*(x,y) \geq 1$ *for all* $x,\ y \in X$ *and* $d^*(x,y) = 1$ *if and only if* $x = y$,

$(m2^*)$ $d^*(x,y) = d^*(y,x)$ *for all* $x,\ y \in X$,

$(m3^*)$ $d^*(x,z) \leq d^*(x,y)d^*(y,z)$ *for all* $x,\ y,\ z \in X$ *(multiplicative triangle inequality).*

If operator d^* *satisfies* $(m1^*) - (m3^*)$ *then the pair* (X, d^*) *is called a multiplicative metric space.*

The previous definition was motivation for a large number of papers where the authors proved various fixed point theorems for different contraction conditions in mentioned space (see for example [1]-[4], [9], [16], [19]-[26]).

The next definition for metric spaces is well known:

Definition 2.1.2 *Let* $X \neq \emptyset$*. An operator* $d : X \times X \to \mathbb{R}$ *is a metric on* X*, if it satisfies:*

1. $d(x,y) \geq 0$ *for all* $x,\ ;\ y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

2. $d(x,y) = d(y,x)$ *for all* $x,\ y \in X$,

3. $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x,\ y,\ z \in X$ *(standard triangle inequality).*

If operator d *satisfies* $(1) - (3)$ *then the pair* (X, d) *is called a metric space.*

In ([12]) the following theorem is given.

Theorem 2.1.1 *Let* (X, d^*) *be a multiplicative metric space. Then the pair* (X, d) *is a metric space where* $d(x,y) = \ln d^*(x,y)$ *for all* $x,\ y \in X$*. Conversely, if* (X, d) *is a metric space then* (X, d^*) *is a multiplicative metric space where* $d^*(x,y) = e^{d(x,y)}$ *for all* $x,\ y \in X$*.*

Also, in ([5], [12], [13]) the equivalence between well-known theorems in metric and multiplicative metric spaces has been thoroughly analyzed (Banach [8], Kannan [20], Edelstein-Nemitskii [15], Boyd-Wong [10] and other).

2.2 Main Results

Definition 2.2.1 *([17]) Two self mappings A and S of a multiplicative metric space (X, d^*) are said to be compatible on X if*

$$\lim_{n\to\infty} d^*(ASx_n, SAx_n) = 1$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in \mathcal{X}$.

Definition 2.2.2 *([18]) Two self mappings A and S of a multiplicative metric space (X, d^*) are said to be weakly compatible on X if $Ax = Sx$ for all $x \in X$ implies $ASx = SAx$, that is, $d^*(Ax, Sx) = 1$ i.e. $d^*(ASx, SAx) = 1$.*

Theorem 2.2.1 *([4],) Let (X, d^*) be a complete multiplicative metric space. Let S, T, A, $B : X \to X$ be such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that*

$$\begin{aligned} d^{*p}(Sx, Ty) \;\le\; & \left[\varphi\left(\max\left\{d^{*p}(Ax, By), \frac{d^{*p}(Ax, Sx)d^{*p}(By, Ty)}{1 + d^{*p}(Ax, By)},\right.\right.\right. \\ & \left.\left.\left.\frac{d^{*p}(Ax, Ty)d^{*p}(By, Ax)}{1 + d^{*p}(Ax, By)}\right\}\right)\right]^{\lambda}, \end{aligned} \tag{2.1}$$

for all x, $y \in X$ and $p \ge 1$; where $\varphi : [0, \infty) \to [0, \infty)$ is a monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

Suppose that one of the following conditions is satisfied:

1. *either A or S is continuous, the pair (S, A) is compatible and the pair (T, B)*

is weakly compatible;

2. *either B or T is continuous, the pair (T, B) is compatible and the pair (S, A) is weakly compatible.*

Then S, T, A and B have a unique common fixed point in X.

Remark 2.2.1 *The function φ is superfluous, because $\varphi(t) < t$ and therefore*

$$\begin{aligned} d^{*p}(Sx, Ty) &\leq \left[\varphi\left(\max\left\{d^{*p}(Ax, By), \frac{d^{*p}(Ax, Sx)d^{*p}(By, Ty)}{1 + d^{*p}(Ax, By)},\right.\right.\right. \\ &\left.\left.\left.\frac{d^{*p}(Ax, Ty)d^{*p}(By, Ax)}{1 + d^{*p}(Ax, By)}\right\}\right)\right]^{\lambda} \\ &\leq \left[\max\left\{d^{*p}(Ax, By), \frac{d^{*p}(Ax, Sx)d^{*p}(By, Ty)}{1 + d^{*p}(Ax, By)},\right.\right. \\ &\left.\left.\frac{d^{*p}(Ax, Ty)d^{*p}(By, Ax)}{1 + d^{*p}(Ax, By)}\right\}\right]^{\lambda} \\ &\leq \left[\max\left\{d^{*p}(Ax, By), d^{*p}(Ax, Sx)d^{*p}(By, Ty), d^{*p}(Ax, Ty)\right\}\right]^{\lambda}. \end{aligned}$$

So,

$$d^{*p}(Sx, Ty) \leq \left[\max\left\{d^{*p}(Ax, By), d^{*p}(Ax, Sx)d^{*p}(By, Ty), d^{*p}(Ax, Ty)\right\}\right]^{\lambda}, \quad (2.2)$$

and therefore

$$d^{*}(Sx, Ty) \leq \left[\max\left\{d^{*}(Ax, By), d^{*}(Ax, Sx)d^{*}(By, Ty), d^{*}(Ax, Ty)\right\}\right]^{\lambda}. \quad (2.3)$$

If we apply ln on both sides of (2.3) we get

$$d(Sx, Ty) \leq \lambda \max\left\{d(Ax, By), d(Ax, Sx) + d(By, Ty), d(Ax, Ty)\right\}. \quad (2.4)$$

In the next theorem we prove that condition (2.4) with assumption as in Theorem 2.2.1 provides existence of a common fixed point.

Remark 2.2.2 *In previous theorem, the following condition for the function* φ*:* $\varphi : [1,\infty) \to [1,\infty)$ *is a monotone increasing function such that* $\varphi(1) = 1$ *and* $\varphi(t) < t$ *for all* $t > 0$ *should stay instead of the condition given in theorem.*

Definition 2.2.3 *([17]) Two self-mappings* A *and* S *of a metric space* (X, d) *are called compatible if,*

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ *for some* $t \in X$*. It is easy to see that compatible mappings commute at their coincidence points.*

Definition 2.2.4 *([18]) Two self mappings* A *and* S *of a metric space* (X, d) *are called weakly compatible if, they commute at coincidence points. That is, if* $Ax = Sx$ *implies that* $ASx = SAx$ *for* $x \in X$*, i.e.* $d(ASx, SAx) = 0$*.*

Theorem 2.2.2 *Let* (X, d) *be a complete metric space. Let* S*,* T*,* A*,* $B : X \to X$ *be such that* $S(X) \subset B(X)$*,* $T(X) \subset A(X)$ *and there exists* $\lambda \in \left(0, \frac{1}{2}\right)$ *such that (2.4) is satisfied for all* x*,* $y \in X$*.*

Suppose that one of the following conditions is satisfied:

(a) *either* A *or* S *is continuous, the pair* (S, A) *is compatible and the pair* (T, B) *is weakly compatible;*

(b) *either* B *or* T *is continuous, the pair* (T, B) *is compatible and the pair* (S, A) *is weakly compatible.*

Then S*,* T*,* A *and* B *have a unique common fixed point in* X*.*

Proof. Let $x_0 \in X$. Since $S(X) \subset B(X)$ and $T(X) \subset A(X)$, there exist x_1, $x_2 \in X$ such that $y_0 = Sx_0 = Bx_1$ and $y_1 = Tx_1 = Ax_2$. By induction, we can define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \ \ y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}, \tag{2.5}$$

for all $n \geq 0$: Using (2.4) and (2.5) we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \lambda \max\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad d(Ax_{2n}, Tx_{2n+1})\} \\
&= \lambda \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), \\
&\quad d(y_{2n-1}, y_{2n+1})\} \\
&\leq \lambda \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), \\
&\quad d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} \\
&= \lambda(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})).
\end{aligned}$$

Therefore

$$d(y_{2n}, y_{2n+1}) \leq \frac{\lambda}{1-\lambda} d(y_{2n1}, y_{2n}) = hd(y_{2n1}, y_{2n}). \tag{2.6}$$

Since $\lambda \in \left(0, \frac{1}{2}\right)$ we have that $h \in (0, 1)$.

Also,

$$\begin{aligned}
d(y_{2n+2}, y_{2n+1}) &= d(Sx_{2n+2}, Tx_{2n+1}) \\
&\leq \lambda \max\{d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}) + d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad d(Ax_{2n+2}, Tx_{2n+1})\} \\
&= \lambda \max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1}), \\
&\quad d(y_{2n+1}, y_{2n+1})\} \\
&\leq \lambda \max\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}) + d(y_{2n}, y_{2n+1}), 0\} \\
&= \lambda(d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n+2})),
\end{aligned}$$

and

$$d(y_{2n+2}, y_{2n+1}) \leq \frac{\lambda}{1-\lambda} d(y_{2n+1}, y_{2n}) = hd(y_{2n+1}, y_{2n}). \tag{2.7}$$

Using (2.6) and (2.7) we have that for every $n \in \mathbb{N}$

$$d(y_n, y_{n+1}) \leq hd(y_{n1}, y_n), \;\; h < 1.$$

So, the sequence $\{y_n\}$ is a Cauchy sequence, and since the space is complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$, and since $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are subsequence of $\{y_n\}$ we have

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n+2} = z. \tag{2.8}$$

Suppose that A is continuous. Then $A \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} ASx_{2n} = Az$. Using (2.8) and the assumption that the pair (S, A) is compatible we have that

$$\lim_{n\to\infty}(SAx_{2n}, ASx_{2n}) = \lim_{n\to\infty} d(SAx_{2n}, Az) = 0,$$

which means that $\lim_{n\to\infty} SAx_{2n} = Az$. Using (2.4), we have

$$\begin{aligned} d(SAx_{2n}, Tx_{2n+1}) &\leq \lambda \max\{d(A^2x_{2n}, Bx_{2n+1}), d(A^2x_{2n}, SAx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad d(A^2x_{2n}, Tx_{2n+1})\}. \end{aligned}$$

Taking $n \to \infty$ in the above inequality we have

$$\begin{aligned} d(Az, z) &\leq \lambda \max\{d(Az, z), d(Az, Az) + d(z, z), d(Az, z)\} \\ &= \lambda d(Az, z). \end{aligned}$$

Therefore, $Az = z$. Using again (2.4) we have

$$\begin{aligned} d(Sz, Tx_{2n+1}) &\leq \lambda \max\{d(Az, Bx_{2n+1}), d(Az, Sz) + d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad d(Az, Tx_{2n+1})\}. \end{aligned}$$

Letting $n \to \infty$ we have

$$\begin{aligned} d(Sz, z) &\leq \lambda \max\{d(z, z), d(z, Sz) + d(z, z), d(z, z)\} \\ &= \lambda d(Sz, z), \end{aligned}$$

i.e. $z = Sz = Az$. Since $z = Sz \in S(X) \subset B(X)$, there exist $z_1 \in X$ such that $z = Az = Sz = Bz_1$. Using (2.4) we have

$$\begin{aligned} d(z, Tz_1) = d(Sz, Tz_1) &\leq \lambda \max\{d(Az, Bz_1), d(Az, Sz) + d(Bz_1, Tz_1), d(Az, Tz_1)\} \\ &= \lambda \max\{d(z, z), d(z, z) + d(z, Tz_1), d(z, Tz_1)\} \\ &= \lambda d(z, Tz_1). \end{aligned}$$

Therefore $z = Az = Sz = Bz_1 = Tz_1$. Since the pair T, B weakly compatible, we have $Tz = TBz_1 = BTz_1 = Bz$. It remains to prove that $z = Tz$. Using (2.4) we have

$$\begin{aligned} d(z, Tz) = d(Sz, Tz) &\leq \lambda \max\{d(Az, Bz), d(Az, Sz) + d(Bz, Tz), d(Az, Tz)\} \\ &= \lambda d(z, Tz). \end{aligned}$$

This implies that $z = Tz = Bz = Az = Sz$, and so z is a common fixed point of S, T, A, B. Similarly, if we suppose that S is continuous we have the same conclusion. Next we prove that S, T, A, B have a unique common fixed point. Suppose that u is another common fixed point. Then, using (2.4) we have

$$\begin{aligned} d(z, u) = d(Sz, Tu) &\leq \lambda \max\{d(Az, Bu), d(Az, Sz) + d(Bu, Tu), d(Az, Tu)\} \\ &= \lambda d(z, u), \end{aligned}$$

i.e. $z = u$. ■

Theorem 2.2.3 *Let (X, d^*) be a complete multiplicative metric space. Let S, T, A, $B : X \to X$ be such that $S(X) \subset B(X)$, $T(X) \subset A(X)$ and there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that condition (2.3) is satisfied for all x, $y \in X$.*

Suppose that one of the following conditions is satisfied:

(a) *either A or S is continuous, the pair (S, A) is compatible and the pair (T, B) is weakly compatible;*

(b) *either B or T is continuous, the pair (T, B) is compatible and the pair (S, A) is weakly compatible.*

Then S, T, A and B have a unique common fixed point in X.

Theorem 2.2.4 *Theorem 2.2.2 and Theorem 2.2.3 are equivalent.*

Theorem 2.2.5 *([7]) Let S and T be mappings of a complete multiplicative metric space (X, d^*) into itself satisfying the conditions $S(X) \subset X$, $T(X) \subset X$ and*

$$d^*(Sx, Ty) \leq \left(\max\left\{\frac{d^*(x,Sx)[d^*(y,Sx)+d^*(y,Ty)]}{1+d^*(Sx,Ty)}, \frac{d^*(y,Sx)d^*(x,Ty)+d^*(x,y)d^*(Sx,y)}{d^*(Sx,Ty)+d^*(Sx,y)}, \frac{d^*(x,Sx)d^*(y,Sx)+d^*(x,y)d^*(Sx,Ty)}{d^*(y,Ty)+d^*(y,Sx)}, \frac{d^*(y,Ty)d^*(x,Ty)+d^*(x,Ty)d^*(y,Sx)}{d^*(y,Ty)+d^*(y,Sx)}\right\}\right)^{\lambda}, \tag{2.9}$$

for all $x, y \in X$, where $\lambda \in \left(0, \frac{1}{2}\right)$. Then S and T have a unique common fixed point.

Remark 2.2.3 *Lets look at each member of the right hand side of equation (2.9). Now we have the following.*

$$\begin{aligned}
\frac{d^*(x,Sx)[d^*(y,Sx)+d^*(y,Ty)]}{1+d^*(Sx,Ty)} &\leq d^*(x,Sx)d^*(y,Ty), \\
\frac{d^*(y,Sx)d^*(x,Ty)+d^*(x,y)d^*(Sx,y)}{d^*(Sx,Ty)+d^*(Sx,y)} &\leq d^*(y,Ty)d^*(x,Ty), \\
\frac{d^*(x,Sx)d^*(y,Sx)+d^*(x,y)d^*(Sx,Ty)}{d^*(y,Ty)+d^*(y,Sx)} &\leq d^*(y,Sx)d^*(x,y), \\
\frac{d^*(y,Ty)d^*(x,Ty)+d^*(x,Ty)d^*(y,Sx)}{d^*(y,Ty)+d^*(y,Sx)} &= d^*(x,Ty).
\end{aligned}$$

Our new contractive condition is the following one:

$$d^*(Sx,Ty) \leq (\max\{d^*(x,Sx)d^*(y,Ty), d^*(y,Ty)d^*(x,Ty), d^*(y,Sx)d^*(x,y)d^*(x,Ty)\})^{\lambda}. \tag{2.10}$$

But, we have the following:

$$d^*(Sx,Ty) \leq (\max\{d^*(x,Sx)d^*(y,Ty), d^*(y,Ty)d^*(x,Ty), d^*(y,Sx)d^*(x,y), d^*(x,Ty)\})^{\lambda} \leq \max\{d^*(x,Sx), d^*(y,Ty), d^*(y,Ty), d^*(x,Ty), d^*(y,Sx), d^*(x,y)\}^{2\lambda}. \tag{2.11}$$

If we apply ln on both sides of (2.11) we get

$$d(Sx,Ty) \leq q\max\{d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx), d(x,y)\} \tag{2.12}$$

where $q = 2\lambda$.

The obtained contractive condition is the well known Ćirić strongly-quasi-contraction [11]. It is also well known that for $q = \frac{3}{4}$ mappings S and T do not have a common fixed point. So, additional condition is necessary. One possible solution is given in the paper [7] where the following definition is given:

Definition 2.2.5 *A pair* $\{S,T\}$ *of a mapping is asymptotically regular at* x_0 *if* $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$ *where* $Sx_{2n} = x_{2n+1}$ *and* $Tx_{2n+1} = x_{2n+2}$, $n \in \mathbb{N}$.

In the same paper the following theorem was proved:

Theorem 2.2.6 *Let* S *and* T *be mappings of a complete metric space* (X,d) *into itself satisfying condition (2.12). Suppose that the pair* $\{S,T\}$ *asymptotically regular at* x_0. *Then* S *and* T *have a common fixed point.*

2.3 Conclusion

Multiplicative metric space was introduced by Bashirov in 2008. After that, a huge number of paper appeared where authors use a various contractive condition used in order to prove a fixed point theorem. But, in the paper [12] on Multiplicative metric space, the authors proved that various well known fixed point theorems in multiplicative metric spaces have equivalent fixed point theorem in metric space. So, natural question has appeared: Is the multiplicative metric space a generalization of the metric space? Based on that, we started to study fixed point theorems in multiplicative metric space where the contractive condition is complicated (i.e. rational type contractive condition) and at first, we conclude that there is not always equivalent theorem in metric space. We analyzed two fixed point theorems in multiplicative metric space. In the first theorem we have shown that we can find a better condition in metric space for which function has a fixed point. We proved that (2.1)$\Rightarrow$(2.3)$\Leftrightarrow$(2.4). So, we get better results in metric space than the ones presented in Theorem 2.2.1. Finally, in the second theorem we found better contractive condition for which function has a fixed point but we assume one additional condition. Open question is the following one: Is it possible to find a better condition in metric space without additional conditions? If answer is negative, we realize that in some cases multiplicative metric space is useful.

Chapter 3

Unique Common Fixed Points in Multiplicative Metric Spaces

Abstract

The main purpose of this chapter is the existence and uniqueness of common fixed points for two pairs of occasionally weakly compatible mappings given in [6], which is more general than compatible and weakly compatible mappings, in multiplicative metric spaces. Our results improve and extend the results of the previous chapter.

Keywords: Multiplicative Metric Spaces, Unique Common Fixed Points, Occasionally Weakly Compatible Mappings.

3.1 Introduction and Preliminaries

Recently in 2008, a novel notion was introduced by Al-Thagafi and Shahzad [6] in order to find unique common fixed points under minimum conditions.

Definition 3.1.1 *([6]) Two self-mappings P and Q of a set X are occasionally weakly compatible if and only if, there is a point ν in X such that $P\nu = Q\nu$ implies*

$PQ\nu = QP\nu$.

Note that, the above concept is more general than the compatibility and the weak compatibility. The following example justifies.

Example 3.1.1 *Let* $X = (0, +\infty)$ *with the multiplicative metric* $d(x, y) = e^{|x-y|}$. *Define* $Y,\ Z : X \to X$ *by*

$$Yx = \begin{cases} 2x^2 & \text{if } x \in (0, 2] \\ \dfrac{32}{x} & \text{if } x \in (2, +\infty), \end{cases} \qquad Zx = \begin{cases} 5x - 2 & \text{if } x \in (0, 2] \\ 2x & \text{if } x \in (2, +\infty). \end{cases}$$

We have $Yx = Zx$ *if and only if* $x = \frac{1}{2}$ *or* $x = 2$ *or* $x = 4$ *and*

$$YZ\frac{1}{2} = \frac{1}{2} = ZY\frac{1}{2};$$

i.e., the pair $\{Y, Z\}$ *is occasionally weakly compatible.*

However,

$$YZ2 = Y8 = 4 \neq 16 = Z8 = ZY2,$$

then, Y *and* Z *are not weakly compatible.*

Also, we have

$$YZ4 = Y8 = 4 \neq 16 = Z8 = ZY8,$$

then, Y *and* Z *are not weakly compatible.*

Now, take the sequence $x_n = 2 - \frac{1}{n}$ *for* $n = 1, 2, \ldots$. *We have*

$$Yx_n = 2x_n^2 \to 8 \text{ when } n \to +\infty,$$

$$Zx_n = 5x_n - 2 \to 8 \text{ when } n \to +\infty,$$

and

$$\begin{aligned} d(YZx_n, ZYx_n) &= d(Y(5x_n - 2), Z(2x_n^2)) = d\left(\frac{32}{5x_n - 2}, 4x_n^2\right) \\ &= e^{\left|\frac{32}{5x_n-2} - 4x_n^2\right|} \to e^{12} \neq 1 \text{ when } n \to +\infty, \end{aligned}$$

that is, Y *and* Z *are not compatible.*

3.2 Unique Common Fixed Points for Two Pairs of Mappings

Theorem 3.2.1 *Let* (X, d) *be a multiplicative metric space. Let* F, G, H *and* K : $X \to X$ *be four mappings satisfying:*

1. F *and* H *are occasionally weakly compatible,*
2. G *and* K *are occasionally weakly compatible,*
3. *for all* x, $y \in X$,

$$\begin{aligned} d(Fx, Gy) \leq\ & (\max\{d(Hx, Ky), d(Fx, Hx).d(Gy, Ky), \\ & d(Hx, Gy), d(Fx, Ky)\})^\lambda, \end{aligned}$$

where $\lambda \in (0, 1)$.

Then, mappings F, G, H *and* K *have a unique common fixed point.*

Proof. According to the first and second conditions, as mappings F and H as well as G and K are occasionally weakly compatible, there exist two elements a and b in X such that $Fa = Ha$ (respectively $Gb = Kb$) implies

$$FHa = HFa$$
$$GKb = KGb,$$

respectively.

To prove the existence and uniqueness of the common fixed point, we need four steps.

<u>**Existence of the common fixed point:**</u>

<u>**Step one:**</u> We claim that $Fa = Gb$. Suppose that we have the contrary, then

$$\begin{aligned} d(Fa, Gb) &\leq (\max\{d(Ha, Kb), d(Fa, Ha).d(Gb, Kb), d(Ha, Gb), \\ &\quad d(Fa, Kb)\})^{\lambda} \\ &= \max\{d(Fa, Gb), d(Fa, Fa).d(Gb, Gb), d(Fa, Gb), \\ &\quad d(Fa, Gb)\})^{\lambda} \\ &= \max\{d(Fa, Gb), 1, d(Fa, Gb), d(Fa, Gb)\})^{\lambda} \\ &= \max\{d(Fa, Gb), 1\})^{\lambda} \\ &= d^{\lambda}(Fa, Gb) \\ &< d(Fa, Gb), \end{aligned}$$

a contradiction, hence, $Fa = Gb$.

Second step: Now, assume that $FFa \neq Fa$, then, we have

$$\begin{aligned} d(FFa, Fa) = d(FFa, Gb) &\leq (\max\{d(HFa, Kb), d(FFa, HFa).d(Gb, Kb), \\ &\quad d(HFa, Gb), d(FFa, Kb)\})^{\lambda} \\ &= (\max\{d(HFa, Kb), d(FFa, HFa), d(HFa, Gb), \\ &\quad d(FFa, Kb)\})^{\lambda} \\ &= (\max\{d(HFa, Fa), d(FFa, HFa), d(HFa, Fa), \\ &\quad d(FFa, Fa)\})^{\lambda}, \end{aligned}$$

using the relationship between mappings F and H, we get

$$\begin{aligned} d(FFa, Fa) &\leq (\max\{d(FHa, Fa), d(FFa, FHa), d(FHa, Fa), \\ &\quad d(FFa, Fa)\})^{\lambda} \\ &= (\max\{d(FFa, Fa), d(FFa, FFa), d(FFa, Fa), \\ &\quad d(FFa, Fa)\})^{\lambda} \\ &= (\max\{d(FFa, Fa), 1\})^{\lambda} \\ &= d^{\lambda}(FFa, Fa) \\ &< d(FFa, Fa), \end{aligned}$$

which is a contradiction, thus, $FFa = Fa$ and consequently $HFa = Fa$.

Third step: By the same manner, suppose that $GGb \neq Gb$, then, we have

$$\begin{aligned} d(Fa, GGb) &\leq (\max\{d(Ha, KGb), d(Fa, Ha).d(GGb, KGb), \\ &\quad d(Ha, GGb), d(Fa, KGb)\})^{\lambda} \\ &= (\max\{d(Ha, KGb), d(GGb, KGb), d(Ha, GGb), \\ &\quad d(Fa, KGb)\})^{\lambda}; \end{aligned}$$

that is,

$$\begin{aligned} d(Gb, GGb) &\le (\max\{d(Gb, KGb), d(GGb, KGb), d(Gb, GGb), \\ &\quad d(Gb, KGb)\})^{\lambda}; \end{aligned}$$

since mappings G and K are occasionally weakly compatible, we get

$$\begin{aligned} d(Gb, GGb) &\le (\max\{d(Gb, GKb), d(GGb, GKb), d(Gb, GGb), \\ &\quad d(Gb, GKb)\})^{\lambda} \\ &= (\max\{d(Gb, GGb), d(GGb, GGb), d(Gb, GGb), \\ &\quad d(Gb, GGb)\})^{\lambda} \\ &= (\max\{d(Gb, GGb), 1\})^{\lambda} \\ &= d^{\lambda}(GGb, Gb) \\ &< d(GGb, Gb), \end{aligned}$$

a contradiction which implies that $GGb = Gb$ and consequently $KGb = Gb$. Therefore, $Fa = Ha = Gb = Kb = z$ is a common fixed point of the four mappings.

Uniqueness of the common fixed point:

Fourth step: Assume the existence of another common fixed point (say t), then,

$$\begin{aligned} d(z, t) = d(Fz, Gt) &\le (\max\{d(Hz, Kt), d(Fz, Hz).d(Gt, Kt), d(Hz, Gt), \\ &\quad d(Fz, Kt)\})^{\lambda} \\ &= (\max\{d(z, t), d(z, z).d(t, t), d(z, t), d(z, t)\})^{\lambda} \\ &= (\max\{d(z, t), 1\})^{\lambda} \\ &= d^{\lambda}(z, t) \\ &< d(z, t), \end{aligned}$$

a contradiction, hence $t = z$. ■

Now, we give an illustrative example which supports our result.

Example 3.2.1 *Endowed* $X = (0, +\infty)$ *with the multiplicative metric* $d(x, y) = e^{|x-y|}$ *and define*

$$Fx = \begin{cases} \frac{x+9}{10} \text{ if } x \in (0,1) \\ 1 \text{ if } x \in [1, +\infty), \end{cases} \quad Gx = \begin{cases} \frac{x+8}{9} \text{ if } x \in (0,1) \\ 1 \text{ if } x \in [1, +\infty), \end{cases}$$

$$Hx = \begin{cases} 100 \text{ if } x \in (0,1) \\ \frac{1}{x} \text{ if } x \in [1, +\infty), \end{cases} \quad Kx = \begin{cases} 1000 \text{ if } x \in (0,1) \\ \frac{1}{x} \text{ if } x \in [1, +\infty). \end{cases}$$

First it is clear to see that F *and* H *are occasionally weakly compatible and* G *and* K *are occasionally weakly compatible. Take* $\lambda = \frac{3}{4}$, *we get*

<u>First case:</u> *For* x, $y \in (0, 1)$, *we have* $Fx = \frac{x+9}{10}$, $Gy = \frac{y+8}{9}$, $Hx = 100$, $Ky = 1000$ *and*

$$\begin{aligned} d(Fx, Gy) &= e^{\left|\frac{x+9}{10} - \frac{y+8}{9}\right|} \\ &\leq \max\left\{e^{900}, e^{\left|100 - \frac{x+9}{10}\right|} \times e^{\left|\frac{y+8}{9} - 1000\right|}, e^{\left|100 - \frac{y+8}{9}\right|}, e^{\left|\frac{x+9}{10} - 1000\right|}\right\}^{\frac{3}{4}} \\ &= (\max\{d(Hx, Ky), d(Fx, Hx).d(Gy, Ky), d(Fx, Ky), d(Hx, Gy)\})^{\lambda}. \end{aligned}$$

<u>Second case:</u> *For* x, $y \in [1, +\infty)$, *we have* $Fx = 1$, $Gy = 1$, $Hx = \dfrac{1}{x}$, $Ky = \dfrac{1}{y}$ *and*

$$\begin{aligned} d(Fx, Gy) &= 1 \\ &\leq \max\left\{e^{\left|\frac{1}{x} - \frac{1}{y}\right|}, e^{\left|\frac{1}{x} - 1\right|} \times e^{\left|1 - \frac{1}{y}\right|}, e^{\left|\frac{1}{x} - 1\right|}, e^{\left|1 - \frac{1}{y}\right|}\right\}^{\frac{3}{4}} \\ &= (\max\{d(Hx, Ky), d(Fx, Hx).d(Gy, Ky), d(Fx, Ky), d(Hx, Gy)\})^{\lambda}. \end{aligned}$$

<u>Third case:</u> *For* $x \in (0, 1)$, $y \in [1, +\infty)$, *we have* $Fx = \frac{x+9}{10}$, $Gy = 1$, $Hx = 100$,

$Ky = \dfrac{1}{y}$ *and*

$$\begin{array}{rcl} d(Fx,Gy) & = & e^{\left|\frac{x+9}{10}-1\right|} \\ & \leq & \max\left\{e^{\left|100-\frac{1}{y}\right|}, e^{\left|100-\frac{x+9}{10}\right|} \times e^{\left|1-\frac{1}{y}\right|}, e^{99}, e^{\left|\frac{x+9}{10}-\frac{1}{y}\right|}\right\}^{\frac{3}{4}} \\ & = & (\max\{d(Hx,Ky), d(Fx,Hx).d(Gy,Ky), d(Fx,Ky), d(Hx,Gy)\})^{\lambda}. \end{array}$$

Fourth case: *Finally, for* $x \in [1,+\infty)$, $y \in (0,1)$, *we have* $Fx = 1$, $Gy = \frac{y+8}{9}$, $Hx = \dfrac{1}{x}$, $Ky = 1000$ *and*

$$\begin{array}{rcl} d(Fx,Gy) & = & e^{\left|1-\frac{y+8}{9}\right|} \\ & \leq & \max\left\{e^{\left|\frac{1}{x}-1000\right|}, e^{\left|\frac{1}{x}-1\right|} \times e^{\left|\frac{y+8}{9}-1000\right|}, e^{\left|\frac{1}{x}-\frac{y+8}{9}\right|}, e^{999}\right\}^{\frac{3}{4}} \\ & = & (\max\{d(Hx,Ky), d(Fx,Hx).d(Gy,Ky), d(Fx,Ky), d(Hx,Gy)\})^{\lambda}. \end{array}$$

So, all the hypotheses of the above theorem are satisfied and 1 *is the unique common fixed point of mappings* F, G, H *and* K.

Conclusion

In this dissertation, we presented some very famous fixed-point theorems such as; the contraction mapping theorem of Banach which is considered as an important tool in the theory of metric spaces, the Brouwer fixed-point theorem which is a fundamental result in topology, it proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces, the Kakutanis theorem which extends the Brouwers one to set-valued functions, the Schauder fixed-point theorem which is an extension of the Brouwers one to topological vector spaces, which may be of infinite dimension. Further, we presented a work about the study of the fixed point theorems with contractions of rational type in multiplicative metric spaces. We analyzed whether it was possible to get better results in the context of metric spaces. In the last chapter, we could improve the main results of the previous chapter by presenting our proper theorem, that is, we could find a common fixed point for two pairs, using only the concept of occasionally weakly compatible mappings which is more general than compatible and weakly compatible notions, in other words, we removed the completeness and the inclusions and the continuity, of course, we furnished an illustrative example to show the validity and credibility of our result.

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