الجمهورية الجزائرية الديمقراطية الشعبية
République Algérienne Démocratique et Populaire
وزارة التعليم العالي والبحث العلمي
Ministry of Higher Education and Scientific Research
جامعة الشادلي بنجديد- الطارف
University Chadli Bendjedid – El Tarf
كلية العلوم و التكنولوجيا
Faculty des Sciences and Technology
قسم الرياضيات
Department de Mathematics

End of studies dissertation

In onder to obtaining the Master's degree

Domain : Mathematics and computer science

Branch: Mathematics

Speciality: Functional Analysis and stochastic calculus

Theme

Study of differential operator in Sobolev spaces.

Presented by: Khiari Nesrien

In front of the Jury :

Dr	Amina Bounouala	**MCB**	Univ Chadli bendjedid El Tarf	Chairwoman
Dr.	Saifia Ouarda	**MCA**	Univ Chadli Bendjedid El Tarf	Co-Supervisor
Dr.	Bouselsal Naila	**MCB**	Univ Chadli Bendjedid El Tarf	Examiner

Academic Year : 2024-2025

Study of a differential operator in Sobolev spaces.

Khiari Nesrine
Departement of Mathecs, University Chadli Bendjedid El-Tarf

Juin 2025

Supervesior : Dr. Bounouala Amina

TABLE DES MATIÈRES

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to Almighty God, who has granted me the health, strength, and perseverance to complete this work. "All praise and thanks be to God."

I extend my sincere appreciation to Mrs. Bounaouala Amina for her invaluable guidance, insightful advice, and unwavering encouragement. Thank you for dedicating your time to support me throughout this academic year and for guiding me in my research.

I am also profoundly grateful to all the professors and teachers who have contributed to my education and intellectual growth throughout my academic journey.

My deepest and most heartfelt thanks go to my parents for their constant support, sacrifices, and encouragement during challenging times and throughout my life.

I would also like to express my utmost gratitude to my dear uncle for his lifelong support, care, and kindness, which I will never forget.

Lastly, I extend my sincere thanks to everyone who, directly or indirectly, contributed to the completion of this work.

Dedication

First, I give thanks to Almighty God for granting me the health, perseverance, and strength to become who I am today and to complete this modest work.

To my life's guiding light, my father AbdeRrazek - my steadfast anchor who planted within me an unyielding ambition that continues to propel me forward. Thank you for devoting your life to our family and for instilling in us a lifelong love of learning. May God grant you long life, my noble father, and may He preserve you for me always.

To the wellspring of love, my dearest mother Amel - whom I treasure above myself, my sanctuary and strength. Your tender heart and ceaseless prayers have been the unseen foundation of my success, and your unwavering faith in me my greatest blessing. No words could ever capture your true worth, my "angel mother."

To my precious sister Narimane, the very pulse of my being who has walked beside me through every trial and triumph - may God protect you always and shower your life with boundless joy and contentment.

To my eldest brother Zidane - my tireless supporter who stood by me through this entire endeavor - may God reward you abundantly. And to my younger brother Abde El Djalil, whose youthful spirit has been a constant source of light and laughter in my life - your presence alone brings peace to my soul.

I must also express my profound gratitude to my uncle Djilani, who shaped my education from childhood, particularly in mastering the French language. His persistent encouragement challenged me to excel, and his steadfast belief in my potential has been instrumental to my academic journey. I remain eternally grateful for his guidance in pushing me toward excellence.

To my dear colleagues - may God grant you the same guidance and success He has bestowed upon my sister. To all my family, both maternal and paternal - thank you for your endless encouragement and support. You have my deepest appreciation.

Résumé

Dans ce travail, nous étudions un problème du type p—Laplacian de la forme :

$$-A_p u + m(x)\,|u|^{p-2}\,u = f\,(x,u) \text{ dans } R^N. \quad (1)$$

Nous montrons l'existence et l'unicité d'une solution faible de (1) dans R^n, en utilisant le théorème de Browder.

Mots clés : l'opérateur p—Laplacien, solution faible, espace de Lebesgue-Sobolev, théorème de Browder.

Abstract

In This work, we study The p—Laplacian problem of The forme :

$$-\Delta_p u + m(x)\,|u|^{p-2}\,u = f\,(x,u) \text{ in } \mathbb{R}^N.$$

We establish the existence and uniqueness of a weak solution of the problem in $\mathbb{R}^n$, which involves the p—laplacian through the Browder Theorem.
Key Words :

p—Laplacian operator, weak solution, Lebesgue-Sobolev space, Brawder Theorem.

1 Notation

$a.e$: Almost everywhere.

R : Real field.

R^N : Euclidean space of dimension N, where N is a nonzero natural number.

▲ : Open set of R^n

ð : The gradient.

A : The la placian.

div : The divergence.

PDE : Partial differential equations.

$L^p(▲)$: The space of p-integrable funtion.

$W^{1,p}(▲)$: Standart Sobolev space on ▲ with exponent p.

$W_0^{1,p}(▲)$: of $D(▲)$ in $W^{1,p}(▲)$ with respect to the norm$\| . \|_{W^{1,p}(▲)}$.

x : Vector in R^n, $x = (x, x, ..., x)$, $x \in \mathrm{R}$, $1 \leq i \leq N$.

CAR : Carathéodory.

2 Introduction

The study of nonlinear partial differential equations (PDEs) occupies a central position in modern mathematical analysis. Among these, equations involving the p-Laplacian operator defined for $1 < p < \infty$ by :

$$\Delta_p u := ð \cdot \ |ðu|^{p-2}\, ðu$$

which represents a nonlinear generalization of the classical Laplace operator (when $p = 2$). This operator is degenerate or singular, which introduces technical challenges in both the analytical and numerical treatment of related problems.

In this thesis, we investigate nonlinear elliptic problems of the form :

$$-\Delta_p u + m(x)\, |u|^{p-2}\, u = f(x, u), \quad x \in \mathrm{R}^N \tag{2}$$

posed in the entire Euclidean space R^N, where $N \geq 1$, $m(x)$ is a positive function, and $f : \mathrm{R}^N \times \mathrm{R} \to \mathrm{R}$ is a Carathéodory function satisfying appropriate growth and monotonicity conditions.

The problems arises the $p-$Laplacian begin in the first half of the $20tk$ century when researchers observed that linear equations failed to accurately model certain physical phenomena characterized by nonlinear behavior. Nonlinear boundary value problems with p-Laplacian operator occur in a variety of physical phenomena, for instance, non-Newtonian fluids, reaction–diffusion problems, petroleum extraction, flow through porous media.

Due to the nonlinear nature of the $p-$Laplacian equation, its analysis requires a combination of advanced mathematical techniqurs. The most notable among them include :

- Functional analysis : Employing Sobolev spaces, particularly $W^{1,p}(\blacktriangle)$, to study solution properties.
- Variational methods : Reformulating the problem as an energy minimization (or maximization) problem, and applying calculus of variations to establish existence results.

- Sub and super-solution methods : Constructing bounding functions to guarantee existence via comparison principles.
- Nonlinear spectral analysis : Examining eigenvalue problems associated with the p-Laplacian.

In [9], the autor established the existence of weak solutions for the p-Laplacian problem with Dirichlet boundary condition

$$\begin{cases} —\Delta_p u = f(x, u) \ \text{ in } \blacktriangle \\ u = 0 \text{ in } 6\blacktriangle \end{cases}$$

where ▲ is a bounded domain with smooth boundary $6\blacktriangle$, using critical point theory.

In [2], existence and uniqueness results were established for a nonlinear boundary value problem involving the weighted p-Laplacian operator in a bounded domain

$$\begin{cases} —\Delta_{a,p} u = f(x, u) \ \text{ in } \blacktriangle \\ u = 0 \text{ in } 6\blacktriangle. \end{cases}$$

The proof relies on variational principles and the representation properties of the associated function spaces.

We study the problem (2) in more general seting than [9] where ▲ is a unbounded domain, and the functional setting is the standard Sobolev space $W^{1,p}(\mathrm{R}^N)$. The unbounded nature of the domain leads to a loss of compactness, necessitating the application of the Browder theorem.

In the squeal, we recal some basic definition and notations, in chapter 2 we show the main results of this thesis.

CHAPITRE 1

Prelimenairy

1 Lebesgues espaces $L^p(\blacktriangle)$

Definition 1.1 *Get* (E, A, μ) *be a measure space,* $1 \leq p < +\infty$*, and* f *a measurable function. We say that*

$$f \in L^p(\blacktriangle) = L^p(E, A, \mu) \quad \text{if} \quad \int_{\blacktriangle} |f|^p d\mu < +\infty$$

fhe norm is defined as :

$$\| f \|_{L^p(\blacktriangle)} = \left(\int_{\blacktriangle} |f|^p d\mu \right)^{\frac{1}{p}}$$

For $p = 1$ *, then* f *is integrable.*

Definition 1.2 *Get* (E, A, μ) *be a measure space and* $1 \leq p < +\infty$.

For $5 \in L^p_{\mathbb{R}}(E, A, \mu)$*. me define* $\|5\|_p = \|f\|_p$ *if* $f \in 5$*. We also recall that*

$$5 = f = \{ g \in L^p; \; g = f \text{ almost everymhere} \}$$

fhe space $L^\infty(\blacktriangle)$ is defined as :

$$L^\infty(\blacktriangle) = \{ f : \blacktriangle \to \mathrm{R},\ \textit{measurable},\ \exists\ C > 0 \quad \textit{such that} \quad | f(x) | \leq C \ \textit{a.e on}\ \blacktriangle\}$$

mith the norm :

$$|| f ||_\infty = \inf\{C; | f(x) | \leq C \ \textit{a.e on}\ \blacktriangle\}$$

Theorem 1.1 *(Fricher−Riez)*

fhe space $L^p(\blacktriangle)$ is a Banach space if $1 \leq p \leq +\infty$.

Proof. For $p = +\infty$, let $(f_n)_n$ be a Cauchy sequence in L^∞. Let $h \geq 1$, $\exists\ N_k \in \mathbb{N}$

$$\|f_m - f_n\|_{L^\infty} < \frac{1}{h} \qquad \forall m, n \geq N_k$$

$\exists\ E_k$ negligible.

$$|f_m(x) - f_n(x)| < \frac{1}{h} \qquad \forall m, n \geq N_k \quad ,\forall x \in \blacktriangle \backslash E_k$$

we assume that :$E = \cup_k E_k$(E negligible). Therefore :

$$|f_m(x) - f_n(x)| < \frac{1}{h}, \forall m, n \geq N_k, \forall x \in \blacktriangle \backslash E$$

The sequence$(f_n(x))_n$ is Cauchy in (R) who is complete. Then if $m \to +\infty$ We have

$$|f(x) - f_n(x)| \leq \frac{1}{h} \quad \forall x \in \blacktriangle \backslash E \quad \forall n \geq N_k$$

Thus

$$\|f - f_n\|_\infty < \frac{1}{h} \qquad \forall n \geq N_k$$

We have

$$\begin{aligned} \|f\|_\infty &= \|f - f_n + f_n\|_\infty \\ &\leq \|f - f_n\|_\infty + \|f_n\|_\infty \\ &\leq \|f_n\|_\infty + \frac{1}{h} \end{aligned}$$

Therefore $f \in L^\infty$ and

$$\|f - f_n\|_\infty \to_{n\to+\infty} 0$$

hence

$$f_n \to f \in L^\infty$$

L^∞ is a Banach. Let's suppose that $1 \leq p < +\infty$.

Let $(f_n)_n$ be a Cauchy sequence in L^p. We extract a subsequence of f such that

$$6\ n_k \geq n_{k-1} : \left\| f_{n_{h+1}} - f_{n_h} \right\|_{L_p} \leq \frac{1}{2^k},\ 6n, m \geq n_k$$

We define

$$\begin{aligned} g_n(x) &= \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)| \\ &\leq \sum_{k=1}^{n} \|f_{k+1}(x) - f_k(x)\|_{L^p} \end{aligned}$$

g is increasing and

$$\begin{aligned} \|g_n\|_{L^p} &\leq \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \leq \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \\ &\leq 2 - \left(\frac{1}{2}\right)^n \\ &\leq 2 \end{aligned}$$

Therefore $(g_n) \in L^p$.

$$g_n \to g \qquad \text{et } g \in L^p$$

On the other hand $\forall m, n \geq 2$

$$\begin{aligned}
|f_m(x) - f_n(x)| &\leq \sum_{k=m}^{n-1} |f_{k+1}(x) - f_k(x)| \\
&\leq \sum_{k=1}^{m-1} |f_{k+1}(x) - f_k(x)| - \sum_{k=1}^{n-1} |f_{k+1}(x) - f_k(x)| \\
&\leq g(x) - g_{n-1}(x)
\end{aligned}$$

We have,

$$\begin{aligned}
\|f - f_n\|_{L^p} &\leq \|g - g_{n-1}\|_{L^p} \qquad n \to +\infty \\
\|f - f_n\|_{L^p} &\to 0
\end{aligned}$$

So the sequence (f_n) converges to f . Thus, L^p is a Banach space. ■

Theorem 1.2 *(i) For* $p = 2$; *the space* $L^2(\Omega)$ *is a Hilbert space for the inner product.*

$$(f, g)_{L^2(\Omega)} = \int_{\Omega} f(x).g(x)dx.$$

(ii) For $1 \leq p < +\infty$, *(*$L_p, ||.||_p$*) is separable.*

Lemma 1.1 *(Minkomski's inequality) :*

Get $1 \leq p \leq +\infty$, $f, g \in L^p(\Omega)$, *then* $(f + g) \in L^p$ *and*

$$|| f + g ||_p \leq || f ||_p + || g ||_p$$

Lemma 1.2 *(Hölder inequality) :*

Get $1 \leq p, q \leq +\infty$, *mith* p, q *are conjuguate (i.e.* $\frac{1}{p} + \frac{1}{q} = 1$*). If* $f \in L^p(\Omega)$ *and*

$g \in L^q(\blacktriangle)$, then

$$\int_{\blacktriangle} | f.g | \, dx \leq \left(\int_{\blacktriangle} | f |^p dx \right)^{\frac{1}{p}} \left(\int_{\blacktriangle} | g |^q dx \right)^{\frac{1}{p}}$$
$$\leq \; || f ||_p \,.|| g ||_p$$

fhe case $p = q = 2$ gives the Gauchy−Schmartz inequality

$$\int_{\blacktriangle} | f.g | dx \leq \left(\int_{\blacktriangle} | f |^2 dx \right)^{\frac{1}{2}} \left(\int_{\blacktriangle} | g |^2 dx \right)^{\frac{1}{2}}$$

Remark 1.1 *For $1 \leq p \leq +\infty$, the dual of $L^p(\blacktriangle)$ is $L^q(\blacktriangle)$.*

2 The Sobolev spaces $w^{m,p}(\blacktriangle)$

In this section, we introduce the definition of the Sobolev space $W^{m,p}(\blacktriangle)$ and we establish some basic properties.

2.1 Definitions and properties

Let $\blacktriangle$ c R^n be open, $p \in [1, +\infty)$ and $m \in \mathbb{N}$. The sobolev space $W^{m,p}(\blacktriangle)$ is defined by :

$$W^{m,p}(\blacktriangle) = \{ u \in L^p(\blacktriangle) : D \; u \in L^p(\blacktriangle), 6\alpha \in \mathbb{N}^n; | \alpha | \leq m \},$$

where $D \; u$ denotes the weak derivative.

The space $W^{m,p}(\blacktriangle)$ is equipped with the following norm.

$$|| u ||_{W^{m,p}(\blacktriangle)} = \begin{cases} \left(\sum_{\dagger \; \dagger \leq m} ||D \; u||^p_{L^p(\blacktriangle)} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \max_{\dagger \; \dagger \leq m} ||D \; u||_{L^\infty(\blacktriangle)} & \text{if } p = +\infty. \end{cases}$$

Theorem 2.1 *$W^{m,p}(▲)$ is a Banach space.*

Proof. Let $(u_n)_{n2\mathbb{N}}$ be a Cauchy sequence in $W^{m,p}(▲)$, this means that

$$6o > 0, In_0 \in \mathbb{N}, 6s, q \in \mathbb{N}, \ (s \geq n_0 \text{ and } q \geq n_0) \Rightarrow || u_s - u_q ||_{W^{m,q}(▲)} < o.$$

Therefore

$$6o > 0, In_0 \in \mathbb{N}, 6s, q \in \mathbb{N}, (s \geq n_0 \text{ and } q \geq n_0) \Rightarrow @ \sum_{\dagger \ \dagger \leq m} ||D \ (u_s - uq)||^p_{L^p(▲)} \ 1^{\frac{1}{p}} < o.$$

Then

$$(s \geq n_0 \text{ et } q \geq n_0) \Rightarrow \ \|D \ (u_s - u_q)\|_{L^p(▲)} < o \ , \ 6\alpha \in \mathbb{N}^n, | \ \alpha \ | \leq m.$$

From this we deduce that $(D \ u_n)_{n2\mathbb{N}}$ is a Cauchy sequence in $L^p(▲)$, for any $\alpha \in \mathbb{N}^n$, $| \ \alpha \ | \leq m$.

Since $L^p(▲)$ is complete, we get

$$Iv \ \in L^p(▲) \text{ such that } D \ u_n \to v \ , 6\alpha \in \mathbb{N}^n, | \ \alpha \ | \leq m.$$

In particular,

$$u_n \to v_0 \text{ in } L^p(▲) \text{ as } n \to +\infty.$$

To show that

$$v \ = D \ \ v_0, 6\alpha \in \mathbb{N}^n, | \ \alpha \ | \leq m,$$

From the embedding of $L^p(▲)$ into $L^1_{loc}(▲)$ and the Hölder inequality, we obtain

$$|T_{u_n}(') - T_u(')| \ \leq \int_▲ |u_n(x) - u(x)|| \ '(x) \ | \ dx$$
$$\leq \ ||'||_q||u_n - u||_p, \ 6' \in D(▲),$$

where $q = \frac{p}{p-1}$. Hence $6\alpha \in \mathbb{N}^n$, $|\alpha| \le m$

$$T_{D^a u_n}(') \to T_{u_a}('),\ 6' \in D(▲),\ \text{as } n \to +\infty.$$

since

$$D\ T_{u_n} = T_{D^a u_n},$$

by the uniqueness of the limit in $D^j(▲)$, we can conclude that

$$T_{u_a}(') = D\ T_u = (—1)^{\dagger}\ {}^{\dagger}T_u(D\ ').$$

This proves that

$$D\ u \in L^p(▲),\ 6\alpha \in \mathbb{N}^n,\ |\alpha| \le m \Rightarrow v \quad \in W^{m,p}(▲).$$

Hence $W^{m,p}(▲)$ is a complete space. ■

3 Basic Definition and properties of Sobolev Spaces $W^{1,p}(▲)$

Let $▲ \in \mathrm{R}^N$ be an open set and let $p \in \mathrm{R}$ with $1 \le p \le +\infty$.

Definition 3.1 *fhe space* $W^{1,p}(▲)$ *defined by*

$$W^{1,p}(▲) = \left\{ u \in L^p(▲);\ \mathrm{I}\ g_1, g_2, ..., g_N \in L^p(▲);\ '\ _▲ u \frac{6\ '}{6\ x_i} = —'\ _▲ g_i'\ 6' \in C_c^\infty(▲) \atop 6i = 1, 2, ..., N \right\}$$

me mrite :

$$H^1(▲) = W^{1,2}(▲)$$

For $u \in W^{1,p}(▲)$, *me mrite :*

$$\frac{6u}{6x_i} = g_i, \quad \text{and} \quad ðu = (\frac{6u}{6x_1}, \frac{6u}{6x_2}, ..., \frac{6u}{6x_N}) = \text{grad}\, u.$$

fhe spaces $W^{1,p}(\blacktriangle)$ *is equipped mith the norm :*

$$||\,u\,||_{W^{1,p}(\blacktriangle)} = ||u||^p_{L^p} + \sum_{i=1}^{N} \left\| \frac{6u}{6x_i} \right\|_{L_p},$$

or me mrite the equivalent norm by :

$$\left(||u||^p_{L^p} + \sum_{i=1}^{N} \left\| \frac{6u}{6x_i} \right\|^p_{L^p} \right)^{\frac{1}{p}} \; ; \text{ if } \quad 1 \leq p < +\infty.$$

For $p = 2$. $H^1(\blacktriangle) = W^{1,2}(\blacktriangle)$ *is a Hilbert space mith the inner product :*

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \sum_{i=1}^{N} \left\langle \frac{6u}{6x_i}, \frac{6v}{6x_i} \right\rangle_{L^2}$$

fhe associated norm defined by :

$$||u||_{H^1} = \left(||u||^2_{L^2} + \sum_{i=1}^{N} \left\| \frac{6u}{6x_i} \right\|^2_{L^2} \right)^{\frac{1}{2}}$$

is an equivalent norm.

Proposition 3.1 1. *fhe space* $W^{1,p}(\blacktriangle)$ *is a re exive Banach space for* $1 < p \leq +\infty$, *Separable if* $1 < p < +\infty$.

2. *fhe space* $H^1(\blacktriangle)$ *is a separable Hilbet space.*

Proposition 3.2 *(product Derivative) :*

If $u, v \in W^{1,p}(\blacktriangle) \cap L^\infty(\blacktriangle)$ *mith* $1 \leq p \leq +\infty$. *fhen* $uv \in W^{1,p}(\blacktriangle) \cap L^\infty(\blacktriangle)$ *and :*

$$\frac{6}{6x_i}(uv) = \frac{6u}{6x_i} v + u \frac{6u}{6x_i}, \qquad i = 1, 2, \ldots, N$$

Corollary 3.1 *(Poincaré Inequality) : Suppose that* $\blacktriangle$ *is a bounded open set.*

fhen there exists a constant $C > 0$ *such that*

$$\|u\|_{L^p} \leq C \|ðu\|_{L^p} \qquad 6u \in W_0^{1,p}(▲) \ \ (1 \leq p < +\infty)$$

4 Embedding theorem of Sobolev spaces

4.1 Continuous embedding

Definition 4.1 *Get* ▲ *be a domain in* R^n. *We say that* ▲ *verifies the* $m-$*extension property if there exists a continuous linear operator :*

$$P : W^{m,p}(▲) \to W^{m,p}(\mathrm{R}^n)$$

satisfies

1. $P(u)_{A} = u$, $6u \in W^{m,p}(▲)$
2. *For any*

$$0 \leq h \leq m, \mathrm{I}s_k > 0, ||Pu||_{W^{h,p}(\mathrm{R}^n)} \leq C_k||u||_{W^{h,p}(A)}$$

Theorem 4.1 *Get* $I =]a, b[\ \mathbf{c} \ \mathrm{R}$, *there exists a constant* $C > 0$ *such that :*

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,p}(I)} \qquad 6u \in W^{1,p}(I), \quad 61 \leq p < +\infty$$

Othermise $W^{1,p}(I) \mathrm{c} L^\infty(I)$ *mith continuous embedding.*

Theorem 4.2 *Get* ▲ *be a non−empty open of* R^n *possessing the* $m-$*extension property,* $p \in [1, +\infty)$ *and* $m \in \mathbb{N}$. *fhen the space* $D(\mathrm{R}^n)$ *is dense in* $W^{m,p}(▲)$ *i.e.*

$$6u \in W^{m,p}(▲), \mathrm{I}(u_k)_{k2\mathbb{N}} \in D(\mathrm{R}^n), u = \lim_{k\to+\infty}(u_k\chi_{▲}).$$

4.2 Compact embedding

Definition 4.2 *Gompact injection from X into Y means that the unit ball of X is relatively compact in Y, i.e., the closure of $\overline{B_E(0,1)}$ is compact in Y.*

Lemma 4.1 *Get ▲ be a bounded domain in $\mathbb{R}^N$ then the embedding $W^{1,p}(▲) \hookrightarrow L^q(▲)W$ is compact for every $p \geq 1$ if $q < p^*$.*

5 Sobolev Inequalities

If ▲ has dimension 1, then $W^{1,p}(▲) \subset L^\infty(▲)$ with continuous embedding. For dimensions $N \geq 2$, this inclusion holds only when $p > N$; for $p \leq N$, there exist functions in $W^{1,p}$ that fail to belong to L^∞.

However, a fundamental result due to Sobolev asserts that, if $1 \leq p < N$ then $W^{1,p}(▲) \subset L^{p^*}(▲)$ with continuous injection, where $p^* \in]p, +\infty[$.

Case where $▲ = \mathbb{R}^N$

Theorem 5.1 *(Sobolev, Garliardo, Nirenberg)*

Let $1 \leq p < N$, then

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N) \quad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$$

and there exists a constant $C = C(p, N)$ such that

$$\|u\|_{L^{p^*}} \leq C\,\|\partial u\|_{L^p}, \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

Corollary 5.1 *Get $1 \leq p < N$. fhen*

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall q \in [p, p^*]$$

Corollary 5.2 *(fhe case mhere the limit $p = N$)*

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \qquad \forall q \in [N, +\infty[$$

mith continuous injection.

Theorem 5.2 *(Morrey) : Get $p > N$, then $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ mith continuous injection.*

Corollary 5.3 *Get $m \geq 1$ be an integer and $1 \leq p < +\infty$. We have :*

Exponent p	*Embedding*	*Exponent q*
$\frac{1}{p} - \frac{m}{N} > 0$	$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$	$\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$
$\frac{1}{p} - \frac{m}{N} = 0$	$W^{m,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$	$q \in [p, +\infty[$
$\frac{1}{p} - \frac{m}{N} < 0$	$W^{m,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$	$q = +\infty$

With continuous embedding.

6 Fréchet Differentiability :

Let $\mathcal{F} : X \to Y$ be a mapping between two Banach spaces. We say that $\mathcal{F}$ is Fréchet differentiable at a point $u \in X$

if there exists a bounded linear operator $D\mathcal{F}(u) : X \to Y$ such that :

$$\lim_{\|h\|_X \to 0} \frac{\|\mathcal{F}(u+k) - \mathcal{F}(u) - D\mathcal{F}(u)(k)\|_Y}{\|k\|_E} = 0$$

This means that

$$\mathcal{F}(u+k) \approx \mathcal{F}(u) + D\mathcal{F}(u)(k)$$

For small perturbations k, and the approximation error tends to zero.

6.1 In the context of p-Laplacian problems :

Definition 6.1 *(Associted Nonlinear Operator)*

fhe nonlinear operator A is defined meakly as :

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla v dx$$

7 Monotone operators :

Lemma 7.1 *(Minty is trick). Get X be a Banach space, and let $A : X \to X^*$ be a hemi−continuous,* monotone operator. *fhen :*

(i) *fhe operator A is maximally monotone, meaning that if, for a given $u \in X$ and $b \in X^*$, the inequality*

$$\langle b - Av, u - v\rangle_E \geq 0$$

holds for any $v \in X$, then $Au = b$

(ii) *A satisfies condition (M), i.e., from*

$$\begin{aligned} u_n &\sim u \text{ in } X \quad (n \to \infty) \\ Au_n &\sim b \text{ in } X(n \to \infty) \\ \limsup_{n\to\infty} \langle Au_n, u_n\rangle_E &\leq \langle b, u\rangle_E \end{aligned}$$

it folloms that $Au = b$

(iii) From

$$u_n \sim u \text{ in } X \quad \text{and} \quad Au_n \to b \text{ in } X^* \ (n \to \infty),$$
$$u_n \to u \text{ in } X \quad \text{and} \ Au_n \sim b \text{ in } X^* \ (n \to \infty),$$

it folloms that $Au = b$.

Lemma 7.2 *(Principles of Gonvergence). Get X be a Banach space. fhen :*

(i) If $x_n \sim x$ meakly in X as $n \to \infty$, then there exists a constant s such that $\| x_n \|_E \leq s$ for all $n \in \mathbb{N}$.

(ii) If $x_n \sim x$ in X and $f_n \to f$ in X^* as $n \to \infty$, then $\langle f_n, x_n \rangle_E \to \langle f, x \rangle_E$ as $n \to \infty$.

(iii) If $x_n \to x$ in X and $f_n \sim f$ in X^* as $n \to \infty$, then$\langle f_n, x_n \rangle_E \to \langle f, x \rangle_E$ as $n \to \infty$.

(iv) If X is re exive and the sequence (x_n) is bounded, and if all meakly convergent subsequences of (x_n) converge to the same limit x, then the entire sequence (x_n) converges meakly to x.

Definition 7.1 *Get X be a Banach space, and let*

$A : X \to X^*$ be an operator. fhen, A is defined as folloms.

(i) A is monotone if for all $u, v \in X$, me have

$$\langle Au - Av, u - v \rangle_E \geq 0$$

(ii) A is strictly monotone if for all $u, v \in X$ mith $u \neq v$, me have

$$\langle Au - Av, u - v \rangle_E > 0$$

(iii) A is strongly monotone operator if there exists a constant $s > 0$ such that for all

$u, v \in X$ me have

$$\langle Au - Av, u - v \rangle_E \geq s \| u - v \|_E^2$$

(iv) A is coercive if

$$\lim \langle Au, u \rangle_E = \infty \text{ as } \| u \|_E \to \infty$$

Remark 7.1 (i) if A is strongly monotone $\Rightarrow$ A is strictly monotone $\Rightarrow$ A is monotone.

(ii) If A is strongly monotone, then A is also coercive.

In fact, me have :

$$\langle Au, u \rangle_E = \langle Au - A(0), u \rangle_E + \langle A(0), u \rangle_E \geq s \|u\|_E^2 - \|A(0)\|_E \cdot \|u\|_E$$

and

$$\langle Au, u \rangle_E \geq s \|u\|_E - \|A(0)\|_E \to \infty \text{ for } \|u\|_E \to \infty.$$

Exemple 7.1 (i) For the function $g : \mathrm{R} \to \mathrm{R}$ defined as

$g(u) = |u|^{p-2} u$ for $u \neq 0$, and $g(u) = 0$ for $u = 0$,
me have :

(a) For $p > 1$, g is strictly monotone.

(b) *For* $p \geq 2$, $\langle\, g(u) - g(v), u - v \,\rangle_E \geq s\ |u - v|^p$.

(s) *For* $p = 2$, g *is stongly monotone.*

Definition 7.2 *Get* X *and* Y *be Banach space, and let* $A : X \to Y$ *be an operator. fhen :*

(i) A *is completely continuous if :*

$$u_n \sim u \text{ in } X \text{ as } n \to \infty \Rightarrow Au_n \to Au \text{ in } Y \text{ as } n \to \infty$$

(ii) A *is demicontinuous if :*

$$u_n \to u \text{ in } X \text{ as } n \to \infty \Rightarrow Au_n \sim Au \text{ in } Y \text{ as } n \to \infty$$

(iii) A *is hemicontinuous if* $Y = X^*$ *and for all* $u, v, u \in X$, *the function :*

$$t' \to \langle\, A(u + tv), u \,\rangle_E$$

is continuous in the interval $[0, 1]$

(iv) A *is bounded if* A *maps bounded sets in* X *into bounded sets in* Y.

(v) A *is locally bounded if for each* $u \in X$, *there exists* $o(u) > 0$ *and a constant* $h(u)$. *such that for all* $v \in X$ *mith*

$$\|\ u - v\|_E \leq o(u),$$

me have :

$$\|Av\| \leq h(u).$$

Remark 7.2 *fhe folloming implications hold*

(i) A *is compact* $\Rightarrow$ A *is continuous* $\Rightarrow$ A *is demicontinuous* $\Rightarrow$ A *is hemiconti− nuous.*

(ii) A *is bounded* $\Rightarrow$ A *is locally bounded.*

We now establish several immediate consequences of the preceding definitions.

Lemma 7.3 *Get X be a re exive Banach space, and let $A : X \to X^*$ be an operator then :*

(i) If A is demicontinuous, then A is locally bounded.

(ii) If A is monotone, then A is locally bounded.

(iii) If A is monotone and hemicontinuous, then A is demicontinuous.

Proof. ad(iii) : Proof by contradiction : suppose A is not locally bounded. Then, there exixts $u \in X$ and a sequence$(u_n) \subseteq X$ such that $u_n \to u$ in X as $n \to \infty$ and $||Au_n||_E \to \infty$ as $n \to \infty$. Let us define

$a_n : = (1 + \|Au_n\|_E \ \|u_n - u\|_E)^{-1}$. The monotonicity of A yields that for all $v \in X$, we have :

$$0 \leq \langle Au_n - Av, u_n - v \rangle_E = \langle Au_n - Av, (u_n - u) + (u - v) \rangle_E$$

with the a bove notation, this is equivalent to :

$$\begin{aligned} a_n \langle Au_n, v - u \rangle_E &\leq a_n \langle Au_n, u_n - u \rangle_E - \langle Av, u_n - v \rangle_E \\ &\leq a_n \|Au_n\|_E \ \|u_n - u\|_E + \|Av\|_E \ \|u_n\|_E + \|v\|_E \leq 1 + s(v, u) \end{aligned}$$

where we used $a_n \leq 1$ and the boundedness of the sequence (u_n). If we replace v with $2u - v$ in this calculation, we also get $-a_n \langle Au_n, v - u \rangle_E \leq 1 + s(v, u)$. since v can be any point in X, $u := v - u$ is an arbitrary point in X, and we obtain for all $u \in X$:

$$\sup | \ \langle a_n \ Au_n, u \rangle_E \ | \leq s^J(u, u) < \infty$$

The continuous linear operators $a_n \ Au_n : X \to \mathrm{R}$ are bounded by the above calculation.

The principle of uniform boundedness yields :

$$\sup \|a_n \ Au_n\|_E \ \leq s(u)$$

From this and the definition of a_n, we get :

$$\|Au_n\|_E \leq \frac{s(u)}{a_n} = s(u)\,(1 + \|Au_n\|_E \;\; \|u_n - u\|)\,.$$

Since $\|u_n - u\| \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$:

$$s(u)\,\|u_n - u\|_E < \frac{1}{2}$$

and we obtain :

$$\|Au_n\|_E \leq 2s(u)$$

This, the sequence $\|Au_n\|_E$ is bounded, which contradicts the assumption $\|Au_n\|_E \to \infty$ as $n \to 0$. Therefore,

the claim holds.

ad(iv) : Let$(u_n) \subseteq X$ be a sequence with $u_n \to u$ in X as $n \to \infty$. since A is monotone, implication (iii) implies that A is

locally bounded, and thus,(Au_n) is bounded.

Due to the reflexivity of X, there existe a subsequence (u_{nk}) and an element $b \in X^*$such that $Au_{nk} \sim b$ in X^*as

$h \to \infty$. According to lemma $0, 2(iii)$, we have $Au = b$ ie, $Au_{nk} \sim Au$ in X^* as $h \to \infty$. But every weakly convergent

subsequence of (Au_n) converges weakly to Au, as otherwise there would be a subsequence with

$Au_{nl} \sim s \neq b$ as $l \to \infty$ in X^*. Lemma02(iii) would imply $Au = s$, which contradicts $Au = b$, Thus, lemma02(iv)

provides that the entire sequence (Au_n) converges weakly to $b = Au$, ie, A is demicontinuous. ■

Theorem 7.1 *Get A be a re exive real Banach space. Moreover, let $A : X \to X^*$ be an operator mhich is : bounded, demicontinuous, coercive, and monotone on the space X. fhen, the equation $A(u) = f$ has at least one solution $u \in X$ for each $f \in X^*$. If moreover, A is strictly monotone operator, then the equation (p) has* *precisely one solution* $u \in X$ *for every* $f \in X^*$.

8 The Nemyckii Operator :

We define the Nemyckii operator by :

$$(\mathfrak{F}u)(x) := f\ (x, u(x));$$

where $u = (u_1, ..., u_n)^T$, $u : G \subset \mathbb{R}^N \to \mathbb{R}^n$, with adomain $G \subset \mathbb{R}^N$. Regarding the function $f : G \times \mathbb{R}^n \to \mathbb{R}$, we make the folowing assumptions :

(i) carathéodory condition : $f\ (.;\eta) : x \to f\ (x, \eta)$ is measurable on G for all $\eta \in \mathbb{R}^n$, and $f\ (x;.) : \eta \to f\ (x, \eta)$ is continuous on $\mathbb{R}^n$for almost all $x \in G$.

(ii) Growth condition :

$$|f\ (x, \eta)| \leq |a(x)| + b|\eta_i|^{\frac{p_i}{q}}$$

, where $b > 0$ is a constant, $a \in L^q(G)$, $1 \leq q < \infty$, and $p_i \in [1, \infty)$, $i = 1, ..., n$.

CHAPITRE 2

Existence And Uniqueness Of Weak Solution For p-Laplacian Problem In $\mathbb{R}^N$.

1 Introduction :

In this chapiter, we focus on a class of nonlinear elliptic partial differential equations involving the p-Laplacian operator . These equations have attracted considerable attention in recent decades due to their theoretical richness and wide applicability in various scientific and engineering fields.

In this work, we consider the following nonlinear elliptic problems :

$$-\Delta_p u + m(x)\,|u|^{p-2}\,u = f(x,u), \quad x \in \mathbb{R}^N \tag{2.1}$$

posed in the entire Euclidean space $\mathbb{R}^N$, where $N \geq 1$, $m(x)$ is a positive function, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying appropriate growth and monotonicity conditions.

2 Hypothesis :

We make the following assumptions.

(m_0) $m \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < m(x) < +\infty$

Let $r = \frac{p}{p^*-(q+1)}$ and $p^* = \frac{Np}{N-p}$. There exist $a \in L^{(p^*)'}(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ such that :

(f_1) The function f satisfies

$$|f(x,s)| \leq a(x) + b(x)\,|s|^p$$

where

$$1 < q \leq p - 1$$

(f_2) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a carathéodory (CAR) funtion which is decreasing with respect to the second variable, i.e.,

$$f(x, s_1) \leq f(x, s_2)$$

for a.e. $x \in$ ▲ and $s_1, s_2 \in \mathbb{R}, s_1 \geq s_2$.

The goal of this paper is to prove the following result :

Theorem 2.1 *Assume that (m_0) holds and $f \in CAR(\mathbb{R}^N \times \mathbb{R})$ satisfies (f_1) and (f_2). fhen the problem (P) has a unique meak solution.*

We begin by the definition of the weak solution of probléme (2.1).

Definition 2.1 *We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a meak solution of problem (2.1) if*

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} m(x)\,|u|^{p-2}\, uv dx = \int_{\mathbb{R}^N} f(x,u) v dx$$

for all $v \in W^{1,p}(\mathbb{R}^N)$.

For simplicity let $X = W^{1,p}(\mathbb{R}^N)$. *According to condition* (m_0). *me can introduce a nem norm defined as folloms*

$$\|u\| \overset{@}{=} \left(\int_{\mathbb{R}^N} |\eth u|^p \, dx + \int_{\mathbb{R}^N} m(x) \, |u|^p \, dx \right)^{\frac{1}{p}},$$

Definition 2.2 *Get* K *be a Banach space. An operator* $A : K \to K$ *verifies*

$$\langle Au - Av, u - v \rangle \geq 0 \tag{2.2}$$

for any $u, v \in K$ *is called a monotone operator. An operator* A *is called strictly monotone if for* $u \neq v$ *the strict inequality holds in* ((1)). *An operator* A *is called strongly monotone if there exists* $C > 0$ *such that*

$$\langle Au - Av, u - v \rangle \geq C \|u - v\|^2$$

for any $u, v \in K$.

We define the operator $A : X \to X^*$ *by*

$$A := I - 5$$

mhere the operators I *and* 5 *are defined from* X *into* X^* *as*

$$\langle I(u), v \rangle = \int_{\mathbb{R}^N} |\eth u|^{p-2} \, \eth u \eth v dx + \int_{\mathbb{R}^N} m(x) \, |u|^{p-2} \, uvdx$$

and

$$\langle 5(u), v \rangle = \int_{\mathbb{R}^N} f \, (x, u) v dx$$

for all $u, v \in X$.

By Definition 2.1, the main tool in searching the meak solutions of (2.1) is to finding $u \in X$ *mhich satisfies the operator equation* $Au = 0$.

3 Proof of The Main Result

In order to proof the Main result to apply Browder Theorem we split several lemma.

We denote by C and $C_i, i = 1, 2...$ the general positive constants which are the exact values may change from line to line.

Lemma 3.1 *If $(f_1), (f_2)$ hold, then the operator A is bounded.*

Proof. To prove that A is bounded. We know that the functional.

$$W(u) = \int_{\mathbb{R}^N} \frac{1}{p}(|\eth u|^p + m(x)\,|u|^p)dx$$

is of class C^1(sf. [5]) and I is the derivative operator of W in the weak sense, so it yields I is bounded and continuous. Let $u \in X$, such that $\|u\| < K$.

By defing the norm of $5(u)$ in the dual space X^* and using Hölder's inequality, we obtain

$$\|5(x,u)\|_E = \sup_{\|v\|=1} |\langle 5(x,u), v\rangle|$$

then

$$\langle 5(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v(x)dx$$

It follows from (f_1) that we have

$$|f(x,s)| \leq a(x) + b(x)\,|s|^q$$

$$|\langle 5(u), v\rangle| \leq \int_{\mathbb{R}^N} a(x)\,|v(x)|\,dx + \int_{\mathbb{R}^N} b(x)\,|u(x)^q|\,|v(x)|\,dx$$

$$\|5(x,u)\|_E \leq \sup_{\|v\|=1} \left\{ \left(\int_{\mathbb{R}^N} a(x)^{(p')^r} dx\right)^{\frac{1}{(p')^r}} + \left(\int_{\mathbb{R}^N} |u(x)|^p\, dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} (b(x)v(x))^{\frac{p}{p-q}} dx\right)^{\frac{p-q}{p}} \right\}$$

$$\leq \sup_{\|v\|=1} \left\{ \left(\int_{\mathbb{R}^N} a(x)^{(p')^r} dx\right)^{\frac{1}{(p')^r}} + \left(\int_{\mathbb{R}^N} |u(x)|^p\, dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} b(x)^r\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} |v(x)|^p\, dx\right)^{\frac{1}{p}} \right\}$$

Apply the Hölder's inequality to the first term

$$\int_{\mathbb{R}^N} b(x)|u(x)|^q|v(x)|dx \leq \left(\int_{\mathbb{R}^N} |u(x)|^p\, dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N} (b(x)|v(x)|)^{\frac{p}{p-q}} dx\right)^{\frac{p-q}{p}},$$

Since $r = \frac{p}{p-q-1}$

$$\left(\int_{\mathbb{R}^N} (b(x))^r dx\right)^{\frac{1}{g}} < \infty$$

Then we get

$$\|5(x,u\|_E \leq C_3\|a\|_{(p')^r} + C_4\|b\|_r\|u\|^q.$$

Since $|u| \leq K$, we obtain :

$$\|5(x,u)\|_E \leq C_3\|a\|_{(p')^r} + C_4K^q\|b\|_r.$$

Since both I and 5 are bounded, therefore $A = I - 5$ is bounded as well. ■

Lemma 3.2 *If* $(f_1), (f_2)$ *hold, fhen the operator* A *is continuous.*

Proof. It is well known that the functional

$$W(u) = \int_{\mathbb{R}^N} \frac{1}{p}\left(|\eth u|^p + m(x)|u|^p\right) dx$$

is of class C^1. Since I is the Fréchet derivative of W hence I is continuous. Now we check that 5 is completely continuous that is, if $u_n \sim u$ then $5(u_n) \to 5(u)$ and it is well be done. Let u_n is weakly convergent yo u in X so u_n is bounded in X. Set

$$B_k = \left\{ x \in \mathbb{R}^N : |x| < h \right\},$$

we have

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))\, v dx = \int_{\mathbb{R}^N \setminus B_h} (f(x,u_n) - f(x,u)) v dx + \int_{B_h} (f(x,u_n) - f(x,u))\, v dx \to 0$$

In $\mathbb{R}^N \setminus B_k$, For all $v \in X$ we have

$$\int_{\mathbb{R}^N \setminus B_h} a(x)|v|dx \le \left(\int_{\mathbb{R}^N \setminus B_h} |v|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N \setminus B_h} |a|^{(p')'} dx \right)^{\frac{1}{(p')'}}$$

$$\le C||v|| \left(\int_{\mathbb{R}^N \setminus B_h} |a|^{(p')'} dx \right)^{\frac{1}{(p')'}}$$

Similarly we have,

$$\int_{\mathbb{R}^N \setminus B_h} b(x)|u|^q|v|dx \le \left(\int_{\mathbb{R}^N \setminus B_h} |u|^p \, dx \right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N \setminus B_h} (b(x)|v|)^{\frac{p}{p-q}} dx \right)^{\frac{p-q}{p}}$$

$$\le \left(\int_{\mathbb{R}^N \setminus B_h} |u|^p \right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^N \setminus B_h} |v|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N \setminus B_h} b^r \right)^{\frac{1}{r}}$$

$$\le C||u||^q||v|| \left(\int_{\mathbb{R}^N \setminus B_h} b(x)^r dx \right)^{\frac{1}{r}}$$

According to previous inequalities we have,

$$\int_{\mathbb{R}^N \setminus B_h} (f(x,u_n) - f(x,u))\, v dx \leq C||v|| \left(\int_{\mathbb{R}^N \setminus B_h} b(x)^r dx \right)^{\frac{1}{g}} + ||v|| \left(\int_{\mathbb{R}^N \setminus B_h} |a(x)|^{(p')^r} dx \right)^{\frac{1}{(p')^r}}$$

we have $|b|_{L^g(\mathbb{R}^N \setminus B_h)}$ and $|a(x)|_{(p')^r}$ converges to zero as $n \to +\infty$. which yields that

$$\int_{\mathbb{R}^N \setminus B_h} (f(x,u_n) - f(x,u))\, v dx \to 0$$

for h sufficiently large. Next we proof that for h sufficiently large. The sequence (u_n) is weakly convergent to u in X then (u_n) is bounded in X. so $(\chi_{B_h} u_n)$ is bounded in X. we can deduce that $(\chi_{B_h} u_n) \to (\chi_{B_h} u)$ in $L^q(B_k)$ for all $1 \leq q < p^*$ there exist a subsquence u_n and and $k \in L^q(B_k)$such that

$u_n \to u$ and $|u_n| \leq k$ it follows that

$$f(x,u_n) \to f(x,u) \text{ a.e. in } B_k$$

So 5 is completely continuous and then 5 is continuous.

$$\int_{B_h} f(x,u_n) v dx \to \int_{B_h} f(x,u) v dx$$

So 5 is completely continuous and then 5 is continuous. ■

Lemma 3.3 *If $(M_0), (f_1), (f_2)$ hold, fhen the operator A is Monotone.*

Proof. We prove that A is monotone. We recall the following inequality for $p \geq 2$, $x, y \in \mathbb{R}^N$ (*see* [4])

$$|y|^p \geq |x|^p + p\,|x|^{p-2} x(y-x) + \frac{|y-x|^p}{2^{p-1}-1} \tag{2.3}$$

Let

$$\langle I(u) - I(v), u - v\rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u dx - |\nabla v|^{p-2}\nabla v\right) (\nabla u - \nabla v)\, dx + $$
$$+ \int_{\mathbb{R}^N} m(x)(|u|^{p-2}u - |v|^{p-2}v)\,(u - v)\, dx.$$

Applying The inequality (2.3), we obtain

$$\langle I(u) - I(v), u - v\rangle \geq \frac{2}{p^{2p-1} - 1}\left[\int_{\mathbb{R}^N} |\nabla u - \nabla v|^p dx + \int_{\mathbb{R}^N} m(x)|u - v|^p dx\right] = C_p||u - v||^p. \tag{2.4}$$

Therefore, A is strongly monotone. (see e.g. [7]). Further, since f is decreasing with respect to the second variable,

$$\langle 5(u) - 5(v), u - v\rangle = \int_{\mathbb{R}^N} (f\ (x, u) - f\ (x, u)(u - v)dx \leq 0$$

It follows that A is strongly monotone. ■

Lemma 3.4 *If $(M), (f_1), (f_2)$ hold, then the operator A is coercive.*

Proof. We prove that A is a coercive operator. We have

$$\frac{1}{||u||}\langle Au, u\rangle = \frac{1}{||u||}\left[\int_{\mathbb{R}^N} (|\nabla u|^p + m(x)|u|^p)\, dx - \int_{\mathbb{R}^N} f(x, u)u\,dx\right]$$

·The first integral represents the energy terms related to the differential operator A.

·The second integral involves a nonlinear term from the from the function $f\ (x, u)$.

Using (f_1) we have,

$$f(x, u) \leq a(x)\,|u| + b(x)\,|u|^q$$

and by the Holder inequality, we gat

$$\geq \frac{1}{\|u\|}\left[\|u\|^{p}-\int_{\mathbb{R}^N}\left(a(x)\,|u|+b(x)\,|u|^{q}\right)dx\right]$$

$$\geq \frac{1}{\|u\|}\left(\|u\|^{p}-C_1\,\|u\|_{(p)'}\,\|u\|-C_2^{q+1}\,\|u\|^{q}\,\|u\|_{r}\right).$$

which yields the coercivity of A for $1 < q < p - 1$. In the case when $q = p - 1$, since $X \hookrightarrow L^p(\mathbb{R}^N)$ with continuous
embedding, then by a similar argument to that used in [1], A is coercive. ■

Proof of Theorem 2.1.

From the orevious Lemma, the assumptions of Theorem 3 are fulfilled. Therefore, problem (2.1) has a weak solution. For the uniqueness of weak solution for problem (2.1), suppose that u and v be a weak solutions of (2.1) such that $u \neq v$. By (2.4) it follows that

$$0 = \langle Au - Av, u - v\rangle \geq C_p\,\|u - v\|^{p} \geq 0$$

Then $u = v$ and the proof now is completed .

This solution cannot be trivial provided that we suppose $f(x, 0) \neq 0$, because in this case $A(0) \neq 0$.

4 Conclusion

In this dissertation, we studied a nonlinear differential equation involving the p_Laplacian operator in unbounded domain (▲ = R^N).

After presenting the necessary theoretical of Lebesgue_Sobolev spaces, we were able to prove the existence and uniqueness of a weak solution to the studied equation. Due to the unboundedness of the domain. This absence of compactness require an analytical tools such as Browder's theorem.

BIBLIOGRAPHIE

[1] G ; A, Afrouzi, S. Mahdavi, and Z. Naghizadeh, Existence and uniqueness of solution for p-Laplacian dirichlet problem, Unt. J. Nonlinear Sci., 8(2009), 274-278.

[2] Ewa Szlachtowska, ON THE SOLVABILITY OF DIRICHLET PROBLEM FOR THE WEIGHTED p-LAPLACIAN, Opuscula Mathematica • Vol. 32 • No. 4 • 2012

[3] S. Khafagy, Existence and uniqueness of weak solution for weighted p-Laplacian Dirichlet problem, J. Adv. Res. Dyn. Control Syst., 3(2011), 41-49.

[4] J. Leray, J. L. Lions, Quelques resultats de Visik sur les problems elliptiquess non-lineaires par les methodes de Minty Browder, Bull. Soc. Math. France, 93(1965), 97-107.

[5] P. Lindqvist, On the equation $\mathrm{div}(|\eth u|^{p-2}\eth u) + \lambda\,|u|^{p-2}\,u = 0$, Proc. Amer. Math. Soc., 109(1990), 157-164.

[6] M. Mihãilescu and V. Rãdelescu, A multiplicity for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. Roy. Soc. London Ser. A, 462(2006), 2625-2641.

[7] M. Verid, A.Oarraoui, Existence of a weak solution for p-laplacian problem in R^n (2013) Applied mathematics E.note 13, 228-233.

[8] J. S. W. Wong, On the generalized Emden-Fowler equation, SIAM Rev., 17(1975), 339-360.

[9] Wei Dong a, Jiafa Xu b, Existence of weak solutions for a p-Laplacian problem involving Dirichlet boundary condition, Applied Mathematics and Computation.

[10] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Vol. I, II/ A, II/B, III and IV, Springer Verlag, Berlin, Heidelberg, New York, (1986).

[11] Cour d'analyse fonctionnelle avec 200 ex corigés.

[12] Monotone opetators in nonlinear PDEs , WS2023/2024, Dr. Marvin Fritz, JO-HANNESKE PLER, UNIVERSITY Ling.