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Multipl Solution to Neumann problem with variable exponent

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Dédicace

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ملخص

نتناول في هذا العمل مشكل لابلاسيان ب شروط متغيرة مع شروط حدية من نوع نيومان
تطرقنا الى حل هذا النوع من المشاكل باستعمال نظرية كول و الحل الثاني بواسطة نظرية ايكيلاند
الكلمات المفتاحية : $p(x)$ لابلاسيان , الحد الأدنى العالمي, نظرية الممر الجبلي , الحل الضعيف .

Résumé

Dans ce mémoire, nous considérons un problème aux limites pour l'opérateur de type $p(x)$ -Laplacien sous des conditions aux limites non linéaires de Neumann.

L'objectif de ce travail est démontrer l'existence d'un minimum global de la fonctionnelle l'énergie d'Euler-Lagrange, et un minimum locale. En appliquant le principe variationnelle d'Ekland et théorème du Col.

Mots clés : $p(x)$ -Laplacien, solution faible, théorème du col, minimum global.

Abstract

In this paper we consider a boundary value problem for the $p(x)$ –Laplacien operator under nonlinear Neuman type boundray condition .

We establish the existence of a global minimum for the Euler-Lagrange energy. A second weak solution is obtained by the mountain pass theorem.

Key Words : $p(x)$ –Laplacien, weak solution, mountain pass theorem, global minimum.

1 Introduction

The study of nonlinear elliptic equations involving quasilinear homogeneous type operator like the p -Laplace operator is based on the theory of standard Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions.

In the case of non homogeneous $p(\cdot)$ -Laplace operator the natural setting for this approach is the use of the variable exponent Sobolev space the basic idea is to replace the Lebesgue spaces $L^p(\Omega)$ by more general spaces $L^{p(\cdot)}(\Omega)$ called variable exponent Lebesgue spaces.

If the role played by $L^p(\Omega)$ in the definition of the Sobolev spaces $W^{m,p}(\Omega)$ is assigned instead to a variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ the resulting space is denoted by $W^{m,p(\cdot)}(\Omega)$ and called a variable exponent Sobolev space. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev space. The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ were thoroughly studied in [8] and [11].

Variable Sobolev spaces have been used in the last decades to model various phenomena as image restoration and a second major application which uses nonhomogeneous Laplace operator is related to the modelling of electrorheological fluids.

In this paper, we are concerned with so-called $p(x)$ -Laplacian problem with nonlinear Neuman type boundary condition of the form :

$$\begin{cases} -\Delta_{p(x)}u + h(x) |u|^{p(x)-2} u = \lambda g(x) |u|^{q(x)-2} u & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = k(x) |u|^{q(x)-2} u & \text{on } \partial\Omega \end{cases}$$

Where p, q and s are continuous variable exponents. The measurable functions h and g are supposed positive in Ω , the measurable function k is positive in $\partial\Omega$ and λ is a real parameter.

The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the so called $p(x)$ -Laplacian operator and $\frac{\partial}{\partial \nu}$ is the outer unit derivative.

1. Introduction

Many authors have studied the Neuman boundary value problem involving $p(x)$ -Laplacian such as.

In [2] the authors have studied the problem

$$\begin{cases} -\Delta_{p(x)}u + h(x)|u|^{p(x)-2}u = f(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = g(x, u) & \text{on } \partial\Omega \end{cases}$$

They establish the nonexistence, existence and multiplicity of positive solutions using the sub-supersolution principle combined with a variational method.

The author in [12] has investigated the following problem

$$\begin{cases} -\Delta_{p(x)}u + h(x)|u|^{p(x)-2}u = \lambda f(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \mu g(x, u) & \text{on } \partial\Omega. \end{cases}$$

Using Mountain Pass theorem and Fountain theorem they prove the existence of infinite many pairs of weak solutions.

In both works, the authors required a subcritical growth condition on the nonlinearities. Specifically, it was assumed that

$$|f(x, t)| \leq c_1 + c_2 |t|^{q(x)-1}, \forall x \in \Omega, \forall t \in \mathbb{R},$$

$$|g(x, t)| \leq \bar{c}_1 + \bar{c}_2 |t|^{r(x)-1}, \forall x \in \partial\Omega, \forall t \in \mathbb{R},$$

With $q(x) < p^*(x)$ in Ω and $r(x) < p^\partial(x)$ in $\partial\Omega$.

Though the critical point theory, the principal results of this paper are the existence of global and local minimums of the energy functional. Thus we are entitled to say that our problem is case of the mentioned paper since we do not have to saddle the nonlinearities with subcritical condition.

The paper is organised as follows. In chapitre 1, we present some preliminary results on Lebesgue and Sobolev variable exponent spaces. chapitre 2, contains the statements. Finally, we give the proofs in chapitre 2.

CHAPITRE 1

Preliminary and Notation

1 Lebesgue and Sbolev spaces

Definition 1.1 Let X be a $\mathbb{k}$ -vector space. A function $\varrho : X \rightarrow [0, \infty]$ is called a semimodular on X if the following properties hold.

- (a) $\varrho(0) = 0$.
- (b) $\varrho(\lambda x) = \varrho(x)$ for all $x \in X$, $\lambda \in \mathbb{k}$ with $|\lambda| = 1$
- (c) ϱ is convex
- (d) ϱ is left-continuous.
- (e) $\varrho(\lambda x) = 0$ for all $\lambda > 0$ implies $x = 0$.

A semimodular ϱ is called a modular if

- (f) $\varrho(x) = 0$ implies $x = 0$.

A semimodular ϱ is called continuous if

- (g) The mapping $\lambda \rightarrow \varrho(\lambda x)$ is continuous on $[0, \infty)$ for every $x \in X$

Definition 1.2 If ϱ be a semimodular or modular on X , then

$$X_\varrho := \left\{ x \in X : \lim_{\lambda \rightarrow 0} \varrho(\lambda x) \right\} \tag{1.1}$$

Is called a semimodular space or modular space, respectively. The limit $\lambda \rightarrow 0$. takes place in $\mathbb{k}$. Since $\varrho(\lambda x) = \varrho(|\lambda|x)$ it is enough to require $\lim_{\lambda \rightarrow 0} \varrho(\lambda x) \rightarrow \varrho(\lambda x)$ with

$$\lambda \in (0, \infty).$$

Due to

$$\begin{aligned} \varrho(\lambda x) &= \varrho(|\lambda|x) \leq |\lambda| \varrho(x) && \text{for all } |\lambda| \leq 1, \\ \varrho(\lambda x) &= \varrho(|\lambda|x) \geq |\lambda| \varrho(x) && \text{for all } |\lambda| \geq 1. \end{aligned} \quad (1.2)$$

Which says, we can alternatively define X_ϱ by

$$X_\varrho = \{x \in X : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

since for $\lambda' < \lambda$ we have by 1.2 that

$$\varrho(\lambda' x) = \varrho\left(\frac{\lambda'}{\lambda} \lambda x\right) \leq \frac{\lambda'}{\lambda} \varrho(\lambda x) \rightarrow 0 \text{ as } \lambda' \rightarrow 0.$$

Theorem 1.1 *Let ϱ be a semimodular on X . Then X_ϱ is a normed $\mathbb{k}$ -vector space. The norm, called the Luxemburg norm, is defined by*

$$\|x\|_\varrho := \inf \left\{ \lambda > 0 : \varrho\left(\frac{1}{\lambda}x\right) \leq 1 \right\}.$$

Example 1.1 *(Classical Lebesgue spaces). Let $1 \leq p < \infty$. Then the corresponding modular space $(L^0(\Omega))_{\varrho^p}$ coincides with the classical Lebesgue space L^p*

$$\|f\|_p := \|f\|_{\varrho^p} = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}$$

Similarly, the corresponding semimodular space $(L^0(\Omega))_{\varrho^\infty}$ coincides with the classical Lebesgue space L^∞ i.e.

$$\|f\|_\infty := \|f\|_{\varrho^\infty} = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|.$$

1. Lebesgue and Sbolev spaces

The norm $\|\cdot\|_\varrho$ defines our standard topology on X_ϱ . So for $x_\kappa, x \in X_\varrho$ we say that x_κ converges strongly or in norm to x if $\|x_\kappa - x\|_\varrho \rightarrow 0$.

Definition 1.3 Let $p \in P(A, \mu)$ and let either $\varphi_{p(\cdot)} := \widetilde{\varphi}_{p(\cdot)}$ or $\varphi_{p(\cdot)} := \overline{\varphi}_{p(\cdot)}$. Hence we obtain a semimodular :

$$\varrho_{L^{p(\cdot)}(A)}(f) = \int_A \varphi_p(x) (|f(x)|) dx.$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}(A, \mu)$ as the Musielak–Orlicz space $L^{\varphi_{p(\cdot)}}(A, \mu)$ with the norm

$$\|\cdot\|_{L^{p(\cdot)}(A, \mu)} := \|\cdot\|_{L^{\varphi_{p(\cdot)}}(A, \mu)}$$

In particular, the variable exponent Lebesgue space $L^{p(\cdot)}(A, \mu)$ is

$$L^{p(\cdot)}(A, \mu) = \left\{ f \in L^0(A, \mu) : \lim_{\lambda \rightarrow 0} \varrho_{L^{p(\cdot)}(A)}(\lambda f) = 0 \right\}$$

Or equivalently

$$L^{p(\cdot)}(A, \mu) = \left\{ f \in L^0(A, \mu) : \varrho_{L^{p(\cdot)}(A)}(\lambda f) < \infty \text{ for some } \lambda > 0 \right\}$$

Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(A, \mu)} = \inf \left\{ \lambda > 0 : \varrho_{L^{p(\cdot)}(A)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

Note that $\varrho_{L^{p(\cdot)}(A)}$ is a modular if p is finite everywhere. We abbreviate $\varrho_{L^{p(\cdot)}(A)}$ to $\varrho_{p(\cdot)}$ and $\|\cdot\|_{L^{p(\cdot)}(A, \mu)}$ to $\|\cdot\|_{p(\cdot)}$. Moreover, if $\Omega \subset \mathbb{R}^n$ and μ is the Lebesgue measure we simply write $L^{p(\cdot)}(\Omega)$. This definition seems ambiguous, since either

$$\varphi_{p(\cdot)} = \widetilde{\varphi}_{p(\cdot)} \text{ or } \varphi_{p(\cdot)} = \overline{\varphi}_{p(\cdot)}.$$

However, due to lemma 3.1.6 of [3] it is clear that $L^{\tilde{\varphi}_{p(\cdot)}} = L^{\overline{\varphi}_{p(\cdot)}}$ and

$$\|f\|_{\tilde{\varphi}_{p(\cdot)}} \leq \|f\|_{\overline{\varphi}_{p(\cdot)}} \leq \|f\|_{2\tilde{\varphi}_{p(\cdot)}}.$$

Thus, the two definitions agree up to equivalence of norms with constant at most 2. Recall that we have two relevant Φ -functions $\tilde{\varphi}_{p(\cdot)}$ and $\overline{\varphi}_{p(\cdot)}$.

Lemma 1.1 (Norm-modular unit ball property).

If $p \in P(\Omega)$, then $\|f\|_{p(\cdot)} \leq 1$ and $\varrho_{p(\cdot)}(f) \leq 1$ are equivalent. For $f \in L^{p(\cdot)}(\Omega)$ we have

- (a) If $\|f\|_{\rho(\cdot)} \leq 1$ then $\|f\|_{p(\cdot)} \leq \|f\|_{\rho(\cdot)}$.
- (b) If $1 < \|f\|_{p(\cdot)}$, then $\|f\|_{p(\cdot)} \leq \varrho_{p(\cdot)}(f)$.

This lemma follows directly from Lemma and Corollary 2.1.15 of [3].

Theorem 1.2 Let $p \in P(A, \mu)$. Then $\varphi_{p(\cdot)}$ is proper and $L^{p(\cdot)}(A, \mu)$ is proper and $L^{p(\cdot)}(A, \mu)$ is a Banach function space. Its associate space satisfies

$$(L^{p(\cdot)}(A, \mu))' = L^{p'(\cdot)}(A, \mu)$$

and

$$\begin{aligned} \|g\|_{p'(\cdot)} &\leq \|g\|_{(L^{p(\cdot)})'} \leq 2 \|g\|_{p'(\cdot)} \text{ if } \varphi_{p(\cdot)} = \tilde{\varphi}_{p(\cdot)}, \\ \frac{1}{2} \|g\|_{p'(\cdot)} &\leq \|g\|_{(L^{p(\cdot)})'} \leq 2 \|g\|_{p'(\cdot)} \text{ if } \varphi_{p(\cdot)} = \overline{\varphi}_{p(\cdot)}. \end{aligned}$$

For every $g \in L^0(A, \mu)$.

Proof. It follows from Lemma 3.2.11 of [3] that simple functions are contained in $L^{p(\cdot)}$ and $L^{p'(\cdot)}$. Thus $\varphi_{p(\cdot)}$ is proper by Corollary 2.7.9 of [3] and therefore $L^{p(\cdot)}$ is a Banach function space, see also Section 2.7 of [3]. We can apply Theorem 2.7.4 of [3] to $\varphi_{p(\cdot)}$ to get $(L^{p(\cdot)}(A, \mu))' = L^{p'(\cdot)}(A, \mu)$ and

$$\|g\|_{(\varphi_{p(\cdot)})^*} \leq \|g\|_{(L^{p(\cdot)})'} \leq 2 \|g\|_{(\varphi_{p(\cdot)})^*},$$

which is the first estimate of the claim if $\varphi_{p(\cdot)} = \tilde{\varphi}_{p(\cdot)}$, since $(\tilde{\varphi}_{p(\cdot)})^* = \tilde{\varphi}_{p'(\cdot)}$.

Lemma 1.2 (*Hölder's inequality*).

Let $p, q, s \in P(A, \mu)$ be such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}$$

For μ -almost every $y \in A$. Then

$$\varrho_{s(\cdot)}(fg) \leq \varrho_{p(\cdot)}(f) \cdot \varrho_{q(\cdot)}(g), \quad (1.3)$$

$$\|fg\|_{s(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)} \quad (1.4)$$

$$\|fg\|_{s(\cdot)} \leq \left(\left(\frac{s}{p} \right)^+ + \left(\frac{s}{q} \right)^+ \right) \|f\|_{\tilde{\varphi}p(\cdot)} \|g\|_{\tilde{\varphi}q(\cdot)}, \quad (1.5)$$

for all $f \in L^{p(\cdot)}(A, \mu)$ and $g \in L^{q(\cdot)}(A, \mu)$, where in the case $s = p = q = \infty$ we use the convention $\frac{s}{p} = \frac{s}{q} = 1$. In particular, $f, g \in L^{s(\cdot)}(A, \mu)$. If additionally $f \in E^{p(\cdot)}(A, \mu)$ or $g \in E^{q(\cdot)}(A, \mu)$, then $f, g \in E^{s(\cdot)}(A, \mu)$.

Let $f \in L^{p(\cdot)}$ and $g \in L^{q(\cdot)}$. Since f and g are measurable, also fg is measurable. Then (1.3) follows from (3.2.16) of [3], by integration over $y \in A$. The following argument applies to both $\tilde{\varphi}_{p(\cdot)}$ and $\tilde{\varphi}_{q(\cdot)}$.

If $\|f\|_{p(\cdot)} \leq 1$ and $\|g\|_{q(\cdot)} \leq 1$, then $\varrho_{p(\cdot)}(f) \leq 1$ and $\varrho_{q(\cdot)}(g) \leq 1$ by the unit ball property. Using (1.3) we estimate

$$\varrho_{s(\cdot)}\left(\frac{1}{2}fg\right) \leq \frac{1}{2}\varrho_{s(\cdot)}(fg) \leq \frac{1}{2}(\varrho_{p(\cdot)}(f) + \varrho_{q(\cdot)}(g)) \leq 1.$$

This implies $\|fg\|_{s(\cdot)} \leq 2$ by the unit ball property. The scaling argument proves (1.4). Now let $\|f\|_{\tilde{\varphi}p(\cdot)} \leq 1$ and $\|g\|_{\tilde{\varphi}q(\cdot)} \leq 1$ then by the unit ball property $\bar{\varrho}_{p(\cdot)}(f) \leq 1$ and $\bar{\varrho}_{q(\cdot)}(g) \leq 1$.

Using (3.2.17) of [3] integrated over $y \in A$ we get

$$\bar{\varrho}_{s(\cdot)}(ab) \leq \left(\frac{s}{p} \right)^+ \bar{\varrho}_{p(\cdot)}(f) + \left(\frac{s}{q} \right)^+ \bar{\varrho}_{q(\cdot)}(g) \leq \left(\frac{s}{p} \right)^+ + \left(\frac{s}{q} \right)^+.$$

This implies

$$\|fg\|_{\bar{\varphi}^s(\cdot)} \leq \text{esssup} \frac{s}{p} + \text{esssup} \frac{s}{q}$$

by the unit ball property. The scaling argument proves (1.5).

Assume now that additionally $f \in E^{p(\cdot)}$ i.e. $\varrho_{p(\cdot)}(\lambda f) < \infty$ for every $\lambda > 0$. Let $\gamma > 0$ be such that $\varrho_{q(\cdot)}(g/\gamma) < \infty$. Then for every $\lambda > 0$. We have

$$\varrho_{r(\cdot)}(\lambda f g) \leq \varrho_{p(\cdot)}(\lambda \gamma f) + \varrho_{q(\cdot)}(g/\gamma) < \infty.$$

Since $\lambda > 0$ was arbitrary, this proves $f g \in E^{p(\cdot)}$. The case $g \in E^{q(\cdot)}$ follows by symmetry. The case $s = 1$ in Lemma 1.2 is of special interest :

$$\begin{aligned} \int_A |f||g| d\mu &\leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}, \\ \int_A |f||g| d\mu &\leq \left(1 + \frac{1}{p-} - \frac{1}{p+}\right) \|f\|_{\bar{\varphi}^p(\cdot)} \|g\|_{\bar{\varphi}^{p'}(\cdot)} \\ \int_A |f||g| d\mu &\leq \left(1 + \frac{1}{p-} - \frac{1}{p+}\right) \|f\|_{\bar{\varphi}^p(\cdot)} \|g\|_{\bar{\varphi}^{p'}(\cdot)} \end{aligned}$$

for all $f \in L^{p(\cdot)}(A, \mu)$ and $g \in L^{p'(\cdot)}(A, \mu)$. ■

Theorem 1.3 *Let $p, q \in P(A, \mu)$. Define the exponent $r \in P(A, \mu)$ by*

$$\frac{1}{r(y)} := \max \left\{ \frac{1}{q(y)} - \frac{1}{p(y)}, 0 \right\} \text{ for all } y \in A.$$

(a) *If $q \leq p$ μ -almost everywhere and $1 \in L^{r(\cdot)}(A, \mu)$, then $L^{p(\cdot)}(A, \mu) \hookrightarrow L^{p(\cdot)}(A, \mu)$ with norm at most $2 \|1\|_{L^{r(\cdot)}}(A)$.*

(b) *If the measure μ is atom-less and $L^{p(\cdot)}(A, \mu) \hookrightarrow L^{q(\cdot)}(A, \mu)$ with norm $K > 0$, then $q \leq p$ μ -almost everywhere and $\|1\|_{L^{r(\cdot)}}(A) \leq 4K$.*

Lemma 1.3 *Let $p \in P(A, \mu)$ be a bounded exponent and let μ be separable. Then $L^{p(\cdot)}(A, \mu)$ is separable.*

Proof. Since $\varphi_{p(\cdot)}$ is locally integrable by Remark 3.4.3 of [3], we can apply Theorem 2.5.10 of [3] to show that $E^{p(\cdot)}$ is separable. Since $p^+ < \infty$, we further have $E^{p(\cdot)} = L^{p(\cdot)}$ Theoreme 3.4.1 of [3].

■

Theorem 1.4 *Let $p \in P(A, \mu)$ with $1 < p^- \leq p^+ < \infty$. Then $L^{p(\cdot)}(A, \mu)$ is reflexive.*

Proof. Let $1 < p^- \leq p^+ < \infty$. Then it follows from Remark 3.4.3 of [3] that $\varphi_{p(\cdot)}$ and $(\varphi_{p(\cdot)})^*$ are locally integrable. Moreover, by Theorem 3.4.1 of [3] it follows that $E^{p(\cdot)} = L^{p(\cdot)}$ and $E^{p'(\cdot)} = L^{p'(\cdot)}$. Thus Corollary 2.7.18 of [3] shows that $L^{p(\cdot)}$ is reflexive.

■

Next we define the sobolev spaces with variable exponent

Definition 1.4 *Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. If there exists $g \in L^1_{loc}(\Omega)$ such that*

$$\int_{\Omega} u \frac{\partial^{\alpha_1 + \dots + \alpha_n} \psi}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n} dx = (-1)^{\alpha_1 + \dots + \alpha_n} \int_{\Omega} \psi g dx$$

for all $\psi \in C_0^\infty(\Omega)$, then g is called a weak partial derivative of u with respect to α . The function g is denoted by $\partial_\alpha u$ or by $\frac{\partial^{\alpha_1 + \dots + \alpha_n} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$. Moreover, we write ∇u to denote the weak gradient $\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ of u and we write short $\partial_j u$ for $\frac{\partial u}{\partial x_j}$ with

$j = 1, \dots, n$. More generally we write $\nabla^k u$ to denote the btensor with entries $\partial_\alpha u, |\alpha| = k$. If a function u has classical derivatives then they are also weak derivatives of u . Also by definition $\nabla u = 0$ almost everywhere in an open set where $u = 0$.

Definition 1.5 *The function $u \in L^{p(\cdot)}(\Omega)$ belongs to the space $W^{k, p(\cdot)}(\Omega)$, where*

$k \in \mathbb{N}_0$ and $p \in P(\Omega)$, if its weak partial derivatives $\partial_\alpha u$ with $|\alpha| \leq k$ exist and belong to $L^{p(\cdot)}(\Omega)$. We define a semimodular on $W^{k, p(\cdot)}(\Omega)$ by

$$\varrho_{W^{k, p(\cdot)}(\Omega)}(u) := \sum_{0 \leq |\alpha| \leq k} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u)$$

which induces a norm by

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \left\{ \inf_{\lambda > 0} \lambda : \varrho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

For $k \in \mathbb{N}$ the space $W^{k,p(\cdot)}(\Omega)$ is called Sobolev space and its elements are called Sobolev functions. Clearly $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$.

Definition 1.6 Let (A, Σ, μ) un σ fini complet measure space. we ote pat $P(A, \mu)$ the set of μ measurable function $p : A \rightarrow [1, \infty]$.

The fonction $p \in P(A, \mu)$ are called variable exponent on A . We define

$$p^- := p_A^- = \operatorname{ess\,inf}_{y \in A} p(y)$$

and

$$p^+ := p_A^+ = \operatorname{ess\,sup}_{y \in A} p(y)$$

if $p^+ < \infty$, then p is called a boundede variable exponet. For $p \in P(A, \mu)$, we definit $p' \in P(A, \mu)$ by

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \text{ where } \frac{1}{\infty} = 0$$

the function p' is called the conjugue of exposant p .

In the special case μ is the n dimensional lebsgue measure . Ω is a sous an open set of $\mathbb{R}^n$, note $P(\Omega, \mu) = P(\Omega)$ to define the $L^{p(\cdot)}$ space, we need a corresponding generalised φ function.

Definition 1.7 (palais smale condition)

Let X be a banach space and $\Phi : X \rightarrow \mathbb{R}$ a differentiable functional in the gateaux sense. We say that the function Φ satisfies the weak condition of palais smale if for any continuation

$\{x_n\} \subset X$ cheking

$$\sup |\Phi(x_n)| < \infty, \quad \text{and} \quad \lim_{n \rightarrow \infty} \Phi'(x_n) = 0 \in X^*, \forall n \in \mathbb{N}, \Phi(x_n) \neq 0,$$

There is a critical point of $\Phi; \bar{x} \in X$ such as

$$\liminf \Phi(x_n) \leq \Phi(\bar{x}) \leq \limsup \Phi(x_n).$$

Definition 1.8 (weak solution)

The real definition for weak solution for $-\Delta u = 0$ in Ω ; $u = g$ on $\partial\Omega$. $u \in W^{1,2}(\Omega)$ is weak solution if $u = g$ on $\partial\Omega$;

for any $v \in W^{1,2}(\Omega)$, we have

$$\int \nabla u \cdot \nabla v = 0.$$

Theorem 1.5 (intermediate value theoremen)

The intermediate value theorem states the following consider an interval $I = [a, b]$ of real numbers $\mathbb{R}$. Then

i. If u is a number between $f(a)$ and $f(b)$, that there is, $\min(f(a), f(b)) < u < \max(f(a), f(b))$ then there is a $c \in (a, b)$ such that $f(c) = u$.

ii. The set of function value has no gap. For any two function values $c, d \in f(I)$ with $c < d$, even if they are outside the interval between $[c, d]$ are also function values,

$$[c, d] \subseteq f(I)$$

Proposition 1.1 Let $\varrho_{p(x)}(f) = \int_{\Omega} |f|^{p(x)} dx$

(i) For $f \in L^{p(x)}(\Omega)$, we have

$$|f|_{p(x)}^{p^-} \leq \varrho_{p(x)}(f) \leq |f|_{p(x)}^{p^+} \text{ if } |f|_{p(x)} > 1 \quad (1.6)$$

$$|f|_{p(x)}^{p^+} \leq \varrho_{p(x)}(f) \leq |f|_{p(x)}^{p^-} \text{ if } |f|_{p(x)} \leq 1 \quad (1.7)$$

(ii) For $f, f_n \in L^{p(x)}(\Omega)$, we have

$$f_n \rightarrow f \text{ in } L^{p(x)}(\Omega) \text{ if and only if } \varrho_{p(x)}(f_n - f) \rightarrow 0 \quad (1.8)$$

Proof. first we dmonstrate (1.6) suppose $|f|_{p(x)} = a > 1$ then $\varrho\left(\frac{f}{a}\right) = 1$. Note that $\frac{1}{a} < 1$.

if $\lambda > 1$, we have

$$\varrho(f) \leq \lambda \varrho(f) \leq \lambda^{p^-} \varrho(f) \leq \varrho(\lambda f) \leq \lambda^{p^+} \varrho(f).$$

If now $0 < \lambda < 1$, then we have

$$\lambda^{p^+} \varrho(f) \leq \varrho(\lambda f) \leq \lambda^{p^-} \varrho(f) \leq \lambda \varrho(f) \leq \varrho(f).$$

From this followes that

$$\frac{1}{a^{p^+}} \varrho(f) \leq \varrho\left(\frac{f}{a}\right) = 1 \leq \frac{1}{a^{p^-}} \varrho(f)$$

Hence the result ; inequality (1.7) can be demonstrated in a similar way to demonstrate (1.8) ,it's easy to see that f_n converge to f in Ω . $|f_n|^{p(x)}$ converge to $|f|^{p(x)}$ and using the inequality

$$|f_n|^{p(x)} \leq 2^{p^+-1}(|f_n - f|^{p(x)} + |f|^{p(x)})$$

And the vitali integral convergence teoreme , we deduce that $\varrho(f_n) \rightarrow \varrho(f)$.on the other hand, if f_n convergence to f in Ω . we can deduce that $|f_n - f|^{p(x)}$ converge to 0 in Ω . According to the inequality

$$|f_n \rightarrow f|^{p(x)} \leq 2^{p^+-1} \left(|f_n|^{p(x)} + |f|^{p(x)} \right)$$

and by the fact that $\varrho(f_n) \rightarrow \varrho(f)$ we find that

$$||f|^p|_{q(x)} = |f|_{pq}^p.$$

■

Proposition 1.2 *Let p and q be two mesurable functions with $1 \leq p(x)q(x) < \infty$ in Ω . Let $f \in L^{q(x)}(\Omega)$, then*

$$|f|_{p(x)q(x)}^{p^-} \leq \left| |f|^{p(x)} \right|_{q(x)} \leq |f|_{p(x)q(x)}^{p^+} \text{ if } |f|_{p(x)q(x)} > 1, \quad (1.9)$$

$$|f|_{p(x)q(x)}^{p^+} \leq \left| |f|^{p(x)} \right|_{q(x)} \leq |f|_{p(x)q(x)}^{p^-} \text{ if } |f|_{p(x)q(x)} \leq 1, \quad (1.10)$$

In particulier, if $p(x) = q$ is a constant then

$$||f|^p|_{q(x)} = |f|_{pq}^p.$$

Proof. Firstly we demonstrate inequality (1.9).

We know that any function $f \in L^{q(x)}(\Omega)$, satisfies

$$\varrho_{q(x)}(f' |f|_{q(x)}) \leq 1 \quad (1.11)$$

From (1.11) we have

$$\varrho_{p(x)q(x)}(f' |f|_{p(x)q(x)}) \leq 1$$

So

$$\begin{aligned}
 1 &\geq \int_{\{p(x)q(x)<\infty\}} \left(\frac{|f|}{|f|_{p(x)q(x)}} \right)^{p(x)q(x)} dx \\
 &\geq \int_{\{q(x)<\infty\}} \left(\frac{|f|^{p(x)}}{|f|_{p(x)q(x)}^{p(x)}} \right)^{q(x)} dx
 \end{aligned} \tag{1.12}$$

and $\operatorname{ess\,sup}_{p(x)q(x)=\infty} f/|f|_{q(x)} \leq 1$. The last inequality implies that

$$\operatorname{ess\,sup}_{q(x)=\infty} |f|^{p(x)} \leq \operatorname{ess\,sup}_{x \in \Omega} |f|_{p(x)q(x)}^{p(x)}$$

and from (1.12) we have

$$\varrho_{q(x)}(|f|^{p(x)} / \operatorname{ess\,sup}_{x \in \Omega} |f|_{p(x)q(x)}^{p(x)}) \leq 1$$

This proves the first inequality in (1.10) and the second inequality in (1.9).

In a similar way , we have

$$\varrho_{q(x)}(|f|^{p(x)} / \left| |f|^{p(x)} \right|_{q(x)}) \leq 1$$

Hence

$$\begin{aligned}
 1 &\geq \int_{\{p(x)=\infty\}} \left(\frac{|f|^{p(x)}}{\left| |f|^{p(x)} \right|_{q(x)}} \right)^{q(x)} dx \\
 &\geq \int_{\{p(x)q(x)<\infty\}} \left(\frac{|f|}{\operatorname{ess\,sup}_{q(x)} \left| |f|^{p(x)} \right|_{q(x)}^{\frac{1}{p(x)}}} \right)^{p(x)q(x)} dx .
 \end{aligned}$$

We have

$$\operatorname{esssup}_{x \in \Omega} (|f|^{p(x)} / \left| |f|^{p(x)} \right|_{q(x)}) \leq 1$$

We find

$$\operatorname{ess\,sup}_{p(x)q(x)=\infty} |f|^{p(x)} \leq \operatorname{esssup}_{x \in \Omega} \left| |f|^{p(x)} \right|_{q(x)}^{\frac{1}{p(x)}},$$

We have

$$\begin{aligned} \varrho_{p(x)q(x)}(|f| / \operatorname{esssup}_{x \in \Omega} \left| |f|^{p(x)} \right|_{q(x)}^{\frac{1}{p(x)}}) &\leq 1, \\ |f|_{p(x)q(x)} &\leq \operatorname{esssup}_{x \in \Omega} \left| |f|^{p(x)} \right|_{q(x)}^{\frac{1}{p(x)}}. \end{aligned}$$

If $p^- > 0$, this prouve the second inequality in (1.10) and the first inequality (1.9).

If $p^- = 0$ and $\left| |f|^{p(x)} \right|_{q(x)} \leq 1$ then $|f|_{p(x)q(x)} \leq 1$, (≥ 1 also), the first and second inequality in (1.10) are trivial. ■

Remark 1.1 Let u_n, u of $L^{q(\cdot)}(\Omega)$ be such that $u_n \rightarrow u$. By relation 1.8 and holder inequality we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{q(x)} dx = \int_{\Omega} |u|^{q(x)} dx$$

From (1.6) and (1.7) we have

$$\int_{\Omega} |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

Theorem 1.6 Let $p, q \in P(A, \mu)$. The component $r \in P(A, \mu)$ is defined by

$$\frac{1}{r(x)} = \max \left\{ \frac{1}{q(x)} - \frac{1}{p(x)}, 0 \right\} \text{ for all } x \in A$$

A constant at most $2 \|1\|_{L^{r(\cdot)}(A)}$ si $q \leq p$ μ most partout and the $1 \in L^{r(\cdot)}(A)(A, \mu)$

then $L^{p(\cdot)}(A)(A, \mu) \hookrightarrow L^{q(\cdot)}(A)(A, \mu)$, with

Proof. . Given $q \leq p$ presque every where and

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$$

Applying holder's inequality , we find

$$|f|_{q(\cdot)} \leq 2 |1|_{L^{r(\cdot)}(A)} |f|_{p(\cdot)}$$

and this coplets the proof ■

Corollary 1.1 *Let $p, q \in P(A, \mu)$ with $\mu(A) < \infty$. Then*

$$L^{p(\cdot)}(A)(A, \mu) \hookrightarrow L^{q(\cdot)}(A)(A, \mu)$$

and only if $q \leq p$ μ virtualy every where in A , the constant at most 2.

Remark 1.2 *Noting that in $W^{k,p(\cdot)}(\Omega)$*

$$\sum_{m=0}^k \varrho_{L^{p(\cdot)}(\Omega)}(|\nabla^m u|)$$

and

$$\sum_{m=0}^k (|\nabla^m u|)_{L^{p(\cdot)}(\Omega)}$$

Define a semi modular and a note equivalente to the sobolev semi modular and the sobolev nom, respectively.

Theorem 1.7 *Let $p \in P(\Omega)$ The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if*

$$1 < p^- \leq p^+ < \infty.$$

Proof. We proof only the case $k = 1$, the proof for the general case is similar. We first show that the Sobolev spaces is a Banach space; for that let (u_i) be a Cauchy sequence in $W^{k,p(\cdot)}(\Omega)$. We have to show that there exists $u \in W^{1,p(\cdot)}(\Omega)$ such that $u_i \rightarrow u$ in $W^{k,p(\cdot)}(\Omega)$ as $i \rightarrow \infty$. Since the Lebesgue space $L^{p(\cdot)}(\Omega)$ is a Banach space (Theorem 3.2.7) of [3], there exist $u, g_1, \dots, g_n \in L^{p(\cdot)}(\Omega)$. Such that $u_i \rightarrow u$ and $\partial_j u_i \rightarrow g_j$ in $L^{p(\cdot)}(\Omega)$ for every $j = 1, \dots, n$.

Let $\psi \in C_0^\infty(\Omega)$. Since u_i is in $W^{1,p(\cdot)}(\Omega)$. We have

$$\int_{\Omega} u_i \partial_j \psi dx = - \int_{\Omega} \psi \partial_j u_i dx.$$

Strong convergence in $L^{p(\cdot)}(\Omega)$ implies weak convergences and hence we have

$$\int_{\Omega} u_i \partial_j \psi dx \rightarrow \int_{\Omega} u \partial_j \psi dx$$

and

$$\int_{\Omega} \psi \partial_j u_i dx \rightarrow \int_{\Omega} \psi g_j dx$$

As $i \rightarrow \infty$. Thus the right-hand sides on the previous line yield that $(g_1, \dots, g_n)$ is the weak gradient of u . It follows that $u \in W^{1,p(\cdot)}$ and $u_j \rightarrow u$ in $W^{1,p(\cdot)}$.

By Theorem 3.4.4 of [3] $L^{p(\cdot)}(\Omega)$ is separable if $p^+ < \infty$ and by Theorem 3.4.7 of [3], $L^{p(\cdot)}(\Omega)$ is reflexive if $1 < p^- \leq p^+ < \infty$.

By the mapping $u \mapsto (u, \nabla u)$, the space $W^{1,p(\cdot)}(\Omega)$ is a closed subspace of $L^{p(\cdot)}(\Omega) \times (L^{p(\cdot)}(\Omega))^n$. Thus $W^{1,p(\cdot)}(\Omega)$ is separable if $p^+ < \infty$, and reflexive if $1 < p^- \leq p^+ < \infty$ by Proposition 1.4.4 of [3]. For the uniform convexity we note that $W^{1,p(\cdot)}(\Omega)$ satisfies the Δ_2 -condition provided that $p^+ < \infty$. The $L^{p(\cdot)}$ -modular is uniformly convex for $p^- > 1$ by Theorem 2.4.11 of [3] and the proof of Theorem 3.4.9 of [3]. Thus $\varrho_{W^{k,p(\cdot)}}$ is uniform convex as a sum of uniform convex modulars (Lemma 2.4.16) of [3]. Thus $W^{1,p(\cdot)}(\Omega)$ is uniform convex with its own norm by Theorem 2.4.14 of [3]. ■

Theorem 1.8 *Let $p \in P(\mathbb{R}^n)$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if*

$$1 < p^- \leq p^+ < \infty.$$

Proof. Since $W_0^{k,p(\cdot)}(\Omega)$ is a closed subspace of $W^{k,p(\cdot)}(\Omega)$, the properties follow by Proposition 1.4.4 and Theorem 8.1.6 of [3]. ■

Proposition 1.3 *Let $p, q \in P(\Omega) \cap C(\Omega)$ satisfy $p^+ < n$. If $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$, then $q \leq p^*$.*

Proof. Suppose that $q(x) > p^*(x)$ for some $x \in \Omega$. By the continuity of p and q , there exist $s \in (1, n), t \in (1, \infty)$ and $r > 0$ such that

$$p^*(y) < s^* < t < q(y)$$

for every $y \in B(x, r)$. By Corollary 3.3.4 of [3], $W^{1,s}(B(x, r)) \hookrightarrow W^{1,p(\cdot)}(B(x, r))$

and $L^{q(\cdot)}(B(x, r)) \hookrightarrow L^t(B(x, r))$. Since $s^* < t$ we have $W^{1,s}(B(x, r)) \not\hookrightarrow L^t(B(x, r))$.

Thus $W^{1,p(\cdot)}(B(x, r)) \not\hookrightarrow L^{q(\cdot)}(B(x, r))$, which is a contradiction,

and so $q(x) \leq p^*(x)$.

Next we construct a continuous exponent for which the Sobolev embedding $W^{1,p(\cdot)}(\Omega) \rightarrow L^{p^*(\cdot)}(\Omega)$ does not hold. ■

Proposition 1.4 *Let $h \in C(\Omega)$ be such that $0 < h_- \leq h_+ < \infty$. Then the quantity defined by*

$$\|f\|_h = \inf \left\{ \lambda > 0 : \int_{\Omega} \left(\left| \frac{\nabla f}{\lambda} \right|^{p(x)} + h(x) \left| \frac{f}{\lambda} \right|^{p(x)} \right) dx \leq -1 \right\}$$

is an equivalent norm in $W^{1,p(x)}(\Omega)$.

Let us defined the critical Sobolev exponent of p

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

and

$$p^\partial(x) = \left\{ \begin{array}{ll} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) = N. \end{array} \right\}$$

We have the following embedding theorems.

Proposition 1.5 Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $p, q \in C(\overline{\Omega})$. Suppose that $p(x) < N$ and $p(x) \leq q(x) < p^*$; $\forall x \in \overline{\Omega}$.

then there is a compact and continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 1.6 There is a compact and continuous boundary trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ for any $p \in C(\overline{\Omega})$, $p^- > 1$ and any $r \in C(\partial\Omega)$ satisfying $1 \leq r(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

In what follows the letter c will be indiscriminately used to denote various constants when the exact values are irrelevant.

Definition 1.9 Let V be a part of a Banach space X and $F : V \rightarrow \mathbb{R}$. If $u \in V$ is said F to be derivable in the Gateaux sense (or G differentiable in u), if there exists $l \in X'$ such that in any direction z where $F(u + tz)$ exists for $t > 0$ as small enough, the directional $F'_z(u)$ exists and we have :

$$\lim_{t \rightarrow 0^+} \frac{F(u + tz) - F(u)}{t} = \langle l, z \rangle.$$

We will put $F'(u) = l$ and $C^1(X, \mathbb{R})$ or simply $C^1(X)$ denotes the space of all derivative function spaces.

Definition 1.10 (critical point)

Let X is Banach space, $V \subset X$ an open and $F \in C^1(V, \mathbb{R})$. We say that $u \in V$ is a critical point of F if $F'(u) = 0$. If u is not a critical point, we say that u is a regular point of F . If $c \in \mathbb{R}$, we say that c is a critical value of F if there exists $u \in V$ such that $F(u) = c$ and $F'(u) = 0$. If c is not a critical value, that c is a regular value of F .

Definition 1.11 (Carathéodory function)

Let Ω be a bounded open of $\mathbb{R}^N$. a function f from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ is said a caratheodory function if :

- i. The application : $t \rightarrow f(x, t)$ is continuous p.p.x. $\in \Omega$
- ii. The application : $x \rightarrow f(x, t)$ is ,mesurable for all $t \in \mathbb{R}$.

Remark 1.3 The palais smail condition does not prejudge the existence of a critical value . It merely states that if we have such a sequence, it must be relatively compact .To use it effectively in a

useful way, we'll need to be able to demonstrate by another means that such a sequence is exist.

Proposition 1.7 For all $a, b > 0$ and $0 < k < 1$, we have the inequality.

$$at^k - b \leq \left(\frac{a}{b}\right)^{k/(1-k)} , \quad \forall t \geq 0 \quad (1.13)$$

Demonstration. Indeed, the function

$$t \in [0, \infty) \rightarrow t^\theta$$

Is decreasing for all $\theta > 0$, it follows that

$$a - b.t^{1-k} < 0, \quad \forall t > a. \left(\frac{a}{b}\right)^{1/(1-k)}$$

and

$$t^k. (a - b.t^{1-k}) \leq a.t^k < a. \left(\frac{a}{b}\right)^{k/(1-k)} , \quad \forall t \in \left[0, \left(\frac{a}{b}\right)^{1/(1-k)}\right].$$

The last two inequalities show that (1.13) is true.

Proposition 1.8 Let $p \geq 2$, then all for $x_1, x_2 \in \mathbb{R}^N$ we have

$$(|x_1|^{p-2} x_1 - |x_2|^{p-2} x_2).(x_1 - x_2) \geq \frac{2}{p(2^{p-1} - 1)} |x_1 - x_2|^p \quad (1.14)$$

Proof. Of convexity strict $x \rightarrow |x|^p$ implies for any $x_1, x_2 \in \mathbb{R}^N$ we have

$$|x_2|^p > |x_1|^p + p |x_1|^{p-2} x_1(x_2 - x_1) \quad (1.15)$$

Then we eplace x_2 by $\frac{x_1+x_2}{2}$ in (1.14) we find

$$\left| \frac{x_1 + x_2}{2} \right|^p \geq |x_1|^p + \frac{1}{2} p |x_1|^{p-2} x_1(x_2 - x_1) \quad (1.16)$$

Using Clarkson's inequality for $p \geq 2$ such that

$$|x_1|^p + |x_2|^p \geq 2 \left| \frac{x_1 + x_2}{2} \right|^p + 2 \left| \frac{x_1 - x_2}{2} \right|^p \quad (1.17)$$

We arrive at

$$|x_2|^p \geq |x_1|^p + p |x_1|^{p-2} x_1(x_2 - x_1) + 2 \left| \frac{x_1 + x_2}{2} \right|^p \quad (1.18)$$

This is the inequality (1.14) but with the constant 2^{1-p} instead of $\frac{1}{2^{p-1}-1}$. Repeat this procedure , starting with (1.17) and now let's use (1.18) instead of (1.15), we get the constant $2^{1-p} + 4^{1-p}$.

By iteration we finally find the constante

$$2^{1-p} + 4^{1-p} + 8^{1-p} + \dots = \frac{1}{2^{p-1}-1}. \quad \blacksquare$$

Theorem 1.9 (*Eklend's variational principale*)

Let (M, d) be a complet metric espace . Suppose that

$$\Phi : M \rightarrow (-\infty, +\infty), \Phi \neq \infty$$

Is a semi continuous functional bounded from below. Then ,for all $\varepsilon > 0$ and $v \in M$, there existe $u \in M$ such that

- i) $\Phi(u) \leq \Phi(v) - \varepsilon d(u, v)$,
- ii) $\Phi(x) \leq \Phi(u) - \varepsilon d(x, u)$ for all $x \in M$,

Corollary 1.2 Let M be a banach space and le $I \in C^1(M, \mathbb{R})$. We assume that I is bounded from below and satisfies the palais smale condition at the level

$$c = \inf_{x \in M} I(x).$$

Then I reaches its minumum c .

Proposition 1.9 Let X be a reflexive banach space , $K \subset X$ closed convexe and $J : K \rightarrow \mathbb{R}$ a weakly sequential semibcontinuos lowerment function . Moreover, if K is not bounded , we assume that for any sequence $(x_n)_n$ of K such that $\|x_n\| \rightarrow \infty$, we have $J(x_n) \rightarrow +\infty$. Then J is inferiorly bounded and so minimum in other words :

$$\exists u \in K, \quad J(u) = \inf_{v \in K} J(v) = \min_{v \in K} J(v).$$

Theorem 1.10 (Mountain pass theorem)

Let V be a banach space and $J \in C^1(V, \mathbb{R})$ a functional that satisfies the condition of palais smale and if

- (i) $J(0) = 0$
- (ii) There exists $R > 0$ and $a > 0$ such that if $\|u\|_V = R$ then $J(u) \geq a$,
- (iii) There exists $v \in V$, $\|v\|_V > R$ such that $J(v) < a$.Then, J admits a

critical value $c \geq a$. more precisely, if we ask

$$\Gamma = \{\varphi([0, 1]) ; \varphi \in C([0, 1] ; V), \varphi(0) = 0, \quad \varphi(1) = v\}$$

and

$$c = \inf_{A \in \Gamma} \max_{v \in A} J(v)$$

Then c is a critical value of J .

CHAPITRE 2

Multiple solution to a Neumann problem with variable exponent

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$.

This paper deals with weak solution of the following nonlinear boundary value problem

$$\begin{cases} -\Delta_{p(x)}u + h(x) |u|^{p(x)-2}u = \lambda g(x) |u|^{q(x)-2}u & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = k(x) |u|^{q(x)-2}u & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

Where p, q and s are continues variable exponents. The measurable functions h and g are supposed positive in Ω , the measurable function k is positive in $\partial\Omega$ and λ is a real parameter .The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is the so called $p(x)$ -Laplacian operator and $\frac{\partial}{\partial \nu}$ is the outer unit derivative.

Next ; we state some necessary assumption to proove our results.

1 Hypothesis

We assume $p, q, s \in C_+(\overline{\Omega})$ such that

$$(A) \quad 1 < s^- \leq s^+ < q^- \leq q^+ < p^- \leq p^+ < N, \forall x \in \Omega.$$

We suppose that the function g, h and k are positive, $0 < h_- \leq h_+ < \infty$ and that

$$(B) \quad g \in L^{r(x)}(\Omega), \quad h \in C(\Omega) \cap L^{m(x)}(\Omega) \quad \text{and} \quad k \in L^{l(x)}(\partial\Omega)$$

and we suppose that

$$(C) \quad \frac{p^*(x)}{p^*(x) - q(x)} < r(x), \quad \frac{p^*(x)}{p^*(x) - s_1(x)} < m(x), \quad \forall x \in \Omega.$$

$$\frac{p^\partial(y)}{p^\partial(y) - s_2(y)} < l(y), \quad \forall y \in \partial\Omega.$$

Now we present our results.

2 Statements of Results

Definition 2.1 We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of problem 2.1 .if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} h(x) |u|^{p(x)-2} u v dx - \lambda \int_{\Omega} g(x) |u|^{q(x)-2} u v dx - \int_{\partial\Omega} k(x) |u|^{s(x)-2} u v d\sigma = 0$$

For all $v \in W^{1,p(x)}(\Omega)$ where $d\sigma$ is the surface measure on $\partial\Omega$.

Define $I : W^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ by

$$I(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{h(x)}{p(x)} |u|^{p(x)} dx - \lambda \int_{\Omega} \frac{g(x)}{q(x)} |u|^{q(x)} dx - \int_{\partial\Omega} \frac{k(x)}{s(x)} |u|^{s(x)} d\sigma.$$

Then standard arguments [1] show that $I \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and

$$\langle I'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} h(x) |u|^{p(x)-2} u v dx - \lambda \int_{\Omega} g(x) |u|^{q(x)-2} u v dx - \int_{\partial\Omega} k(x) |u|^{s(x)-2} u v d\sigma$$

3. Existence of multiple solution

For all $u, v \in W^{1,p(x)}(\Omega)$.

Then the weak solution of problem 2.1 correspond to critical point of the functional I .

Firstly, we are looking for a global minimum of functional I which is a solution with negative energy.

Theorem 2.1 *For any $\lambda \in \mathbb{R}$ there exists $u_1 \in W^{1,p(x)}(\Omega)$ a nontrivial weak solution of problem 2.1 such that*

$$I(u_1) < 0$$

We establish also the existence of another solution.

Theorem 2.2 *For any $\lambda \in \mathbb{R}$ there exists $u_2 \in W^{1,p(x)}(\Omega)$ a nontrivial weak solution of problem 2.1 such that*

$$I(u_2) > 0$$

The proofs of the above results are the subject of the next section.

3 Existence of multiple solution

We begin this section by proving the first results of our travel.

Proof of Theorem 2.1 Applying the Holder inequality we reach for all $u \in W^{1,p(x)}(\Omega)$

$$\int_{\Omega} g(x) |u|^{q(x)} dx \leq 2 \|g\|_{r(x)} \|u\|_{q(x)r(x)}^{q^i} \quad , \quad i = + \text{ or } -$$

and

$$\int_{\partial\Omega} k(x) |u|^{s(x)} d\sigma \leq 2 \|k\|_{l(x)} \|u\|_{s(x)l(x)}^{s^i} \quad , \quad i = + \text{ or } -$$

Where $\alpha' = \frac{\alpha}{\alpha-1}$.

Assumption (C) ensure that $q(x)r'(x) < p^*(x)$ in Ω and $s(x)l'(x) < p^{\partial}(x)$ in $\partial\Omega$, so by sobolev embidding theoreme there exist

$c_1 > 0$, $c_2 > 0$ such that

$$\int_{\Omega} g(x) |u|^{q(x)} dx \leq c_1 |g|_{r(x)} \left(\|u\|_{1,p(x)}^{q^+} + \|u\|_{1,p(x)}^{q^-} \right) \quad (2.2)$$

and

$$\int_{\partial\Omega} k(x) |u|^{s_2(x)} d\sigma \leq c_2 |k|_{l(x)} \left(\|u\|_{1,p(x)}^{s^+} + \|u\|_{1,p(x)}^{s^-} \right). \quad (2.3)$$

Let us prove that I is bounded from below in $W^{1,p(x)}(\Omega)$.

We recall the fundamental inequality

$$at^\alpha - bt^\beta \leq a \left(\frac{a}{b} \right)^{\alpha/(\beta-\alpha)}; \quad \forall t > 0 \quad (2.4)$$

For any $a, b > 0$ and $0 < \alpha < \beta$.

If $\lambda > 0$ we have by (2.2) and (2.3)

$$\begin{aligned} I(u) &\geq \frac{1}{p^+} \left(\int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} h(x) |u|^{p(x)} dx \right) - \frac{\lambda c_1}{q^-} |g|_{r(x)} \left(\|u\|_{1,p(x)}^{q^+} + \|u\|_{1,p(x)}^{q^-} \right) \\ &\quad - \frac{c_2}{s_2^-} |k|_{l(x)} \left(\|u\|_{1,p(x)}^{s^+} + \|u\|_{1,p(x)}^{s^-} \right). \end{aligned}$$

However by proposition 1.4 there exists $c > 0$ such that

$$\begin{aligned} I(u) &\geq \frac{c}{p^+} |u|_{1,p(x)}^{p^+} - \frac{\lambda c_1}{q^-} |g|_{r(x)} \left(\|u\|_{1,p(x)}^{q^+} + \|u\|_{1,p(x)}^{q^-} \right) \\ &\quad - \frac{c_2}{s_2^-} |k|_{l(x)} \left(\|u\|_{1,p(x)}^{s^+} + \|u\|_{1,p(x)}^{s^-} \right), \quad i = + \text{ or } - \end{aligned}$$

3. Existence of multiple solution

We can decompose the right hand side in the last inequality as follows

$$\begin{aligned}
I(u) \geq & -\left(\frac{\lambda c_1}{q^-} |g|_{r(x)} \|u\|_{1,p(x)}^{q^+} - \frac{c}{4p^+} \|u\|_{1,p(x)}^{p^i}\right) \\
& - \left(\frac{\lambda c_1}{q^-} |g|_{r(x)} \|u\|_{1,p(x)}^{q^-} - \frac{c}{4p^+} \|u\|_{1,p(x)}^{p^i}\right) \\
& - \left(\frac{c_2}{s^-} |k|_{l(x)} \|u\|_{1,p(x)}^{s^+} - \frac{c}{4p^+} \|u\|_{1,p(x)}^{p^i}\right) \\
& - \left(\frac{c_2}{s^-} |k|_{l(x)} \|u\|_{1,p(x)}^{s^+} - \frac{c}{4p^+} \|u\|_{1,p(x)}^{q^i}\right)
\end{aligned}$$

Applying inequality (2.4) four times it yields :

$$I(u) \geq M$$

Where

$$\begin{aligned}
M = & -\frac{\lambda c_1}{q^-} |g|_{r(x)} \left(\left(4 \frac{\lambda c_1}{c q^-} |g|_{r(x)} p^+\right)^{q^+/(p^i-q^+)} + \left(4 \frac{\lambda c_1}{c q^-} |g|_{r(x)} p^+\right)^{q^-/(p^i-q^-)} \right) \\
& - \frac{c_2}{s^-} |k|_{l(x)} \left(\left(4 p^+ \frac{c_2}{c s^-} |k|_{l(x)}\right)^{s^+/(p^i-s^+)} + \left(4 p^+ \frac{c_2}{c s^-} |k|_{l(x)}\right)^{s^-/(p^i-s^-)} \right) < 0.
\end{aligned}$$

Now if $\lambda \leq 0$ we have

$$\begin{aligned}
I(u) \geq & -\left(\frac{c_2}{s^-} |k|_{l(x)} \|u\|_{1,p(x)}^{s^+} - \frac{c}{2p^+} \|u\|_{1,p(x)}^{p^i}\right) \\
& - \left(\frac{c_2}{s^-} |k|_{l(x)} \|u\|_{1,p(x)}^{s^-} - \frac{c}{2p^+} \|u\|_{1,p(x)}^{p^i}\right)
\end{aligned}$$

and we get by inequality (2.4)

$$I(u) \geq M'$$

where

$$M' = -\frac{c_2}{c s^-} |k|_{l(x)} \left(\left(2 p^+ \frac{c_2}{c s^-} |k|_{l(x)}\right)^{s_2^+/(p^i-s_2^+)} + \left(2 p^+ \frac{c_2}{c s^-} |k|_{l(x)}\right)^{s_2^-/(p^i-s_2^-)} \right) < 0.$$

Next, we show that the palais smale condition is fulfilled.

Lemma 3.1 *The functionale I satisfies the palais smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $(u_n) \subset W^{1,p(x)}(\Omega)$ be a palais smale sequence at level $c \in \mathbb{R}$. that is

$$I(u_n) \rightarrow c$$

$$I'(u_n) \rightarrow 0 \text{ in } (W^{1,p(x)}(\Omega))' \text{ as } n \rightarrow +\infty.$$

We show that (u_n) is bounded in $W^{1,p(x)}(\Omega)$. Let us assume, to the contrary, that $\|u_n\|_{1,p(x)} \rightarrow +\infty$ as $n \rightarrow +\infty$. Since $I'(u_n)$ is a convergent sequence in $(W^{1,p(x)}(\Omega))'$ then there exists $c_3 > 0$ such that $\|I'(u_n)\|_* \leq c_3$ where $\|\cdot\|_*$ is the norm in $(W^{1,p(x)}(\Omega))'$. It follow that

$$|\langle I'(u_n), u_n \rangle| \leq \|I'(u_n)\|_* \|u_n\|_{1,p(x)} \leq c_3 \|u_n\|_{1,p(x)}.$$

There also exists $c_4 > 0$ such that $I(u_n) \leq c_4$. For any $\theta \in \mathbb{R}$ such that

$$0 < \theta < \frac{1}{p^+}$$

We have

$$\begin{aligned} c_3 \|u_n\|_{1,p(x)} + c_4 &\geq I(u_n) - \theta \langle I'(u_n), u_n \rangle \geq \left(\frac{1}{p^+} - \theta\right) \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx + \int_{\Omega} h(x) |u_n|^{p(x)} dx \right) \\ &\quad - \lambda \left(\frac{1}{p^-} - \theta\right) \int_{\Omega} g |u_n|^{q(x)} dx - \left(\frac{1}{s^-} - \theta\right) \int_{\partial\Omega} k |u_n|^{s(x)} d\sigma. \end{aligned}$$

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if $\lambda > 0$ we have by relation (2.2) and (2.3)

$$\begin{aligned} c_3 \|u_n\|_{1,p(x)} + c_4 &\geq \left(\frac{1}{p^+} - \theta\right) \|u_n\|_{1,p(x)}^{p^-} - c_1 \lambda \left(\frac{1}{q^-} - \theta\right) |g|_{r(x)} \left(\|u_n\|_{1,p(x)}^{q^+} + \|u_n\|_{1,p(x)}^{q^-}\right) \\ &\quad - c_2 \left(\frac{1}{s^-} - \theta\right) |k|_{l(x)} \left(\|u_n\|_{1,p(x)}^{s^+} + \|u_n\|_{1,p(x)}^{s^-}\right) \end{aligned}$$

By assumption (A) it yields $\lim_{n \rightarrow +\infty} \|u_n\|_{1,p(x)} < +\infty$ which is a contradiction.

$1 \leq \delta(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

Next we will prove that

$$\int_{\Omega} g \left(|u|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u) dx \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Let $\alpha, \beta \in C_+(\overline{\Omega})$ such that

$$\frac{p(x)}{q(x)-1} \leq \alpha(x) < \frac{p^*(x)}{q(x)-1}, \quad p(x) \leq \beta(x) < p^*(x)$$

And

$$\frac{1}{r(x)} + \frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = 1 \text{ for all } x \in \Omega.$$

According to proposition 1.2 we have

$$\left| \int_{\Omega} g \left(|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u) dx \right| \leq |g|_{r(x)} \left(\left| |u_n|^{q(x)-1} \right|_{\alpha(x)} + \left| |u|^{q(x)-1} \right|_{\alpha(x)} \right) |u_n - u|_{\beta(x)}.$$

Thus there exists $c > 0$ such that

$$\left| \int_{\Omega} g \left(|u|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u) dx \right| \leq c |g|_{r(x)} \left(\|u\|_{1,p(x)}^{q^{i-1}} + \|u\|_{1,p(x)}^{q^{i-1}} \right) |u_n - u|_{\beta(x)}.$$

Since u_n is bounded in $W^{1,p(x)}(\Omega)$ and $u_n \rightarrow u$ in $L^{\beta(x)}(\Omega)$ then

$$\int_{\Omega} g \left(|u|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u) dx \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.5)$$

In similar way we can obtain

$$\int_{\Omega} h \left(|u|^{p(x)-2} u_n - |u|^{p(x)-2} u \right) (u_n - u) dx \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.6)$$

And

$$\int_{\partial\Omega} k \left(|u|^{s(x)-2} u_n - |u|^{s(x)-2} u \right) (u_n - u) d\sigma \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.7)$$

It is easy to see that

$$\left\langle I'(u_n) - I'(u), u_n - u \right\rangle \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.8)$$

this together with relation (2.5), (2.6), and (2.7) give

$$\int_{\Omega} \left(|\nabla u|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \cdot (\nabla u_n - \nabla u) dx \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.9)$$

using the following inequality [4]

$$\left(|x_1|^{p-2} x_1 - |x_2|^{p-2} x_2 \right) (x_1 - x_2) dx \geq \frac{2}{p(2^{p-1} - 1)} |x_1 - x_2|^p \quad (2.10)$$

for all $x_1, x_2 \in \mathbb{R}^N$ and any $p \geq 2$, it yields

$$\frac{2}{p + (2^{p-1} - 1)} \int_{\Omega} |\nabla u_n - \nabla u|^{p(x)} dx \leq \int_{\Omega} \left(|\nabla u|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) (\nabla u_n - \nabla u) dx \quad (2.11)$$

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By the same way we get

$$\frac{2h^-}{p + (2^{p^+ - 1} - 1)} \int_{\Omega} |u_n - u|^{p(x)} dx \leq \int_{\Omega} h(x) \left(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right) (u_n - u) dx. \quad (2.12)$$

Now combining (2.6), (2.8), (2.11) and (2.12) we lead to the fact that $\rho_{1,p(x)}(u_n - u) \rightarrow 0$ as $n \rightarrow +\infty$.

Consequently $u_n \rightarrow u$ in $W^{1,p(x)}(\Omega)$ as $n \rightarrow +\infty$. This means that I palais smale condition and the proof of lemma 3.1 is completed .

In order to reach the assertion of theorem 2.1, we put

$$c = \inf_{u \in W^{1,p(x)}(\Omega)} I(u).$$

Let $(u_n) \subset W^{1,p(x)}(\Omega)$ be a minimizing sequence of I . by Ekeland's variationale principle there exists a sequence $(v_n) \subset W^{1,p(x)}(\Omega)$ such that

$$I(v) \rightarrow c, I'(v_n) \rightarrow 0 \text{ and } \|v_n - u_n\|_{1,p(x)} \rightarrow 0.$$

Since I satisfies the palais smale condition there exists a subsequence still denoted (v_n) , that converge strongly to some $u_1 \in W^{1,p(x)}(\Omega)$.this

minimizer is a critical point of I so a weak solution of problem 2.1.

In order to finalize the proof of theoreme 2.1 we will bring out the non triviality of the critical point u_1 . Let $t > 1$ and $u \in W^{1,p(x)}(\Omega)$.with $u \neq 0$ be fixed .We have

$$I(tu) \leq t^{p^+} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + t^{p^+} \int_{\Omega} \frac{h(x)}{p(x)} |u|^{p(x)} dx - t^{q^-} \int_{\Omega} \frac{g(x)}{q(x)} |u|^{q(x)} dx - t^{s^-} \int_{\partial\Omega} \frac{k(x)}{s(x)} |u|^{p(x)} \partial\sigma.$$

Put

$$f(t) = at^{p^+} - bt^{q^-} - ct^{s^-}$$

where the constant a, b and c are given by

$$0 < a = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{h(x)}{p(x)} |u|^{p(x)} dx,$$

$$b = \lambda \int_{\Omega} \frac{g(x)}{q(x)} |u|^{p(x)} dx$$

and

$$0 < c = \int_{\partial\Omega} \frac{k(x)}{s(x)} |u|^{s(x)} \partial\sigma.$$

We have

$$f'(t) = t^{s^- - 1} f_0(t)$$

where

$$f_0(t) = ap^+ t^{p^+ - s^-} - bq^- t^{q^- - s^-} - cs^-.$$

It is clear that $f_0(t) \rightarrow +\infty$ where $t \rightarrow +\infty$. Thus there exists $t_0 > 0$ such that $f_0(t_0) > 0$, since $f_0(0) > 0$, then by the intermediate value theorem there exists $0 < t_1 < t_0$ such that $f_0(t_1) = 0$. If $f'(t) < 0$ in (t_1, t_0) then we put $\bar{t} = t_1$. if not, it exists $t_2 \in (0, t_0)$ such that $f_0(t_2) > 0$. Applying again intermediate value theorem to the function f_0 in the interval $(0, t_2)$ to ensure the existence of a zero that we note t_3 of the function f_0 . if $f'(t) < 0$ in $(0, t_3)$ and $f'(t) > 0$ in (t_3, t_2) then we put $\bar{t} = t_3$. Repeating this processor enough times to get a $t_{2n+1} > 0$ such that $f'(t) < 0$ in $(0, t_{2n+1})$ and $f'(t) > 0$ in (t_{2n+1}, t_{2n}) for some integer n . we put $\bar{t} = t_{2n+1}$.

This means that $\bar{t}$ is a local minimum of f . So $f(\bar{t}) \leq f(0) = 0$ and this implies that

$$I(\bar{t}u) < 0. \tag{2.13}$$

consequently $I(u_1) < 0$ and the proof of theorem 2.1 is completed. ■

Next, we establish the existence of another nontrivial solution of problem 2.1

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Proof. of theoreme 2.2 . To prove the statement of theorem 2.2 . We will apply the mountain pass theoreme 2.2 . We have already show that I satisfies the palais smale condition and by relation (2.13) there exists $u_0 \in W^{1,p(x)}(\Omega)$ such that $I(u_0) < 0$. Thus it only remains to prove that there exists $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \delta$ for $\|u_n\|_{1,p(x)} = \rho$.

Let $u \in W^{1,p(x)}(\Omega)$ with $\|u_n\|_{1,p(x)} > 1$.

If $\lambda > 0$ we have by relation (2.2) and (2.3)

$$I(u) \geq \frac{1}{p^+} \|u_n\|_{1,p(x)}^{p^+} - \frac{\lambda c_1}{q^-} |g|_{r(x)} (\|u_n\|_{1,p(x)}^{q^+} + \|u_n\|_{1,p(x)}^{q^-}) - \frac{c_2}{s^-} |k|_{l(x)} (\|u_n\|_{1,p(x)}^{s^+} + \|u_n\|_{1,p(x)}^{s^-}).$$

Put $\|u_n\|_{1,p(x)} = \rho$ we get

$$I(u) \geq \rho^{q^+} \left(\frac{1}{p^+} \rho^{p^+ - q^+} - \frac{\lambda c_1}{q^-} |g|_{r(x)} (1 + \rho^{q^- - q^+}) - \frac{c_2}{s^-} |k|_{l(x)} (\rho^{s^+ - q^+} + \rho^{s^- - q^+}) \right).$$

It is clear that under assumption (A) the right hand side of the last inequality tends to $+\infty$ when ρ tends to $+\infty$.

Therefore , there exists $\rho_0 > 0$ such that

$$I(u) \geq \delta(\rho_0) > 0 \text{ for } \|u_n\|_{1,p(x)} = \rho_0.$$

Where

$$\delta(\rho) = \delta(\rho) = \frac{1}{p^+} \rho^{p^+} - \frac{\lambda c_1}{q^-} |g|_{r(x)} (\rho^{q^+} + \rho^{q^-}) - \frac{c_2}{s^-} |k|_{l(x)} (\rho^{s^+} + \rho^{s^-}).$$

We get the same condition in the case of $\lambda \leq 0$ with

$$\delta(\rho) = \frac{1}{p^+} \rho^{p^+} - \frac{c_2}{s^-} |k|_{l(x)} (\rho^{s^+} + \rho^{s^-}) > 0.$$

Hence, by the montain pass theoreme it follows that

$$c = \inf_{\phi \in \Gamma} \max_{t \in [0,1]} I(\phi(t))$$

Is a critical value of I where $\Gamma = \{\phi \in C([0, 1], W^{1,p(x)}(\Omega)) : \phi(0) = 0, \phi(1) = t_1 u_1\}$.

The critical point u_2 associated to c is a weak of problem 2.1 this solution is not trivial since

$c = I(u_2) \geq \delta(\rho_0) > 0$ and theoreme 2.1 is proved ■

Conclusion

In this thesis we have studied some aspect of $p(x)$ -laplacian problems with Neumann boundary conditions.

The methods and results presented in this thesis differ from those for the p -Laplacian . We have seen that the spectral theory of these problems was based on critical point theory. The variational methods used guaranteed either the existence of a global minimum, or the existence of a local minimum, or both. We have essentially applied mountain pass theorem theoreme and Eklund's variational principle to establish ours results.

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