People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Chadli Ben Djedid University ElTarf

Faculty of Sciences and Technology

Department of Mathematics

Course

Functional analysis

Produced by: Ouarda Saifia

University year **2024 / 2025**

Contents

Chapter I

Preliminaries

Contents

I.1 General topology

Definition I.1. *Let X be a non-empty set. A class τ of subsets is a topology on X iff τ satisfies the following axioms*

(1) X and $\emptyset$ belong to τ.

(2) The union of any number of sets in τ belongs to τ.

(3) The intersection of any two sets in τ belongs to τ.

The members of τ are then called $\tau-$ open sets, or simply open sets, and X together with τ i.e the pair (X, τ) is called a topological space.

Example I.1. 1. *Consider the following classes of subsets of*

$$X = \{a, b, c, d, e\}$$

and

$$\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$
$$\tau_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}.$$

Observe that τ_1 is a topology but τ_2 is not a topology on X.

2. *Let X be a set. The collection $\{\emptyset, X\}$ is a topology called the in-discrete topology.*

3. *Let X be a set. The collection $P(X)$ of all sets of X is a topology called discrete topology.*

4. *Let X be a set. Let τ be the collection of all subsets of $U \subset X$ such that X/U is finite or equal to X. Then τ is a topology on X called the finite complement topology.*

I.1.1 Subspace

Let A be a nonempty subset of a topological space (X, τ). The class τ_A of all intersection of A with $\tau-$open subsets of X is a topology on A it is called the subspace topology or topology inherited from X.

Example I.2. 1. *Consider the topology*

$$\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}\}$$

on

$$X = \{a, b, c, d, e\},$$

and the subset $A = \{a, d\}$. *The subspace topology is*

$$\tau_A = \{\emptyset, A, \{a\}, \{d\}\}\,.$$

2. *Consider* $[0, 1]$ *as a subspace of* $\mathbb{R}$, *the set* $\left[0, \dfrac{1}{3}\right[$ *is open in* $[0, 1]$ *because we can write*

$$\left[0, \frac{1}{3}\right[= [0, 1] \cap \left]-\frac{1}{3}, \frac{1}{3}\right[.$$

Of course $\left[0, \dfrac{1}{3}\right[$ *is not open in* $\mathbb{R}$.

I.1.2 Comparison of topologies

Definition I.2. *Let* τ *and* τ' *be two topologies on a set* X. *If* $\tau \subset \tau'$, *we say that* τ' *is finer or strong than* τ. *We also say that* τ *is coarser or weaker that* τ'. *We say that* τ *and* τ' *are comparable if either* $\tau \subset \tau'$ *or* $\tau' \subset \tau$.

Example I.3. *Let* $X = \{a, b, c\}$ *and consider the following topologies*

$$\tau_1 = \{\emptyset, X, a, a, b\}\,,$$

$$\tau_2 = \{\emptyset, X, a, b, a, b\}\,,$$

and

$$\tau_3 = \{\emptyset, X, a, c, a, c\}\,.$$

Then τ_2 *is finer than* τ_1. *However* τ_1 *and* τ_3 *are not comparable.*

Proposition I.1. *Let* X *be a set, and let* τ *and* τ' *be two topologies on* X *such that* $\tau \subseteq \tau'$, $K \subseteq X$, (x_n) *be a sequence in* X *and* $f : X \to \mathbb{R}$ *be a function Then, the following statements hold*

1. *If* K *is compact with respect to* τ', *then* K *is compact for* τ. *In the other word, a weaker topology has more compact sets.*
2. *If* K *is connected with respect to* τ', *then* K *is connected for* τ. *In the other terms, a weaker topology has more connected sets.*
3. *If* (x_n) *is convergent for* τ', *then* (x_n) *is convergent for* τ. *Otherwise stated, a weaker topology has more convergent sequences.*
4. *If* f *is continuous for* τ. *then* f *is continuous for* τ'. *In other words, a weaker topology has fewer continuous functions.*

The concepts of compactness, connectedness, convergence, and continuity play a crucial role in modern analysis, as they underpin numerous existence theorems.

I.1.3 Base for the topology

Let (X, τ) be a topological space. A class $\mathcal{B}$ of open subsets of X, i.e $\mathcal{B} \subset \tau$, is a base for the topology τ iff every open set $G \in \tau$ is the union of members of $\mathcal{B}$. Equivalently, the elements of $\mathcal{B}$ serve as the building blocks for constructing the topology.

I.1.4 Closed sets and related concepts

In topology, a closed set is the complement of an open set. More formally, if X is a topological space and $A \subseteq X$, then A is closed if its complement, $X \backslash A$, is open in X. A limit point of a set $A \subseteq X$ is a point $x \in X$ such that every open neighborhood of x contains at least one point of A different from x itself. The closure of a set A, denoted $\overline{A}$, is the smallest closed set that contains A. It consists of all the points in A, along with any limit points of A.

Examples

1. In $\mathbb{R}$ with the standard topology, any closed interval $[a, b]$, the whole space and the empty set are closed.
2. In a discrete topology (where every subset is open), every set is both open and closed.
3. In $\mathbb{R}$, the set of integers $\mathbb{Z}$ is closed because its complement, $\mathbb{R} \backslash \mathbb{Z}$, is open.

I.1.5 Convergent sequences

A sequence $(a_1, a_2, \ldots)$ of points in a topological space X converges to a point $b \in X$ if and only if, for every open set G containing b, there exists a positive integer $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \in G$. This concept is fundamental in analysis and is key to understanding continuity, limits, and various other topics in mathematics.

I.2 Vector space

A **vector space**, also known as a **linear space**, is a core concept in linear algebra. It is defined over a field $\mathbb{K}$ (commonly $\mathbb{R}$ or $\mathbb{C}$) and follows a set of essential properties. Below is an overview of the key principles associated with vector spaces.

Definition I.3. *A* ***vector space*** *(or* ***linear space****) over a field* $\mathbb{K}$ *is a set* V *equipped with:*

- *Vector addition:* $+ : V \times V \to V$*, satisfying:*
  - *Commutativity:* $u + v = v + u$*,*
  - *Associativity:* $(u + v) + w = u + (v + w)$*,*
  - *Additive identity:* $\exists 0 \in V,\ u + 0 = u$*,*
  - *Additive inverse:* $\forall u \in V,\ \exists - u \in V,\ u + (-u) = 0$*.*
- *Scalar multiplication:* $\cdot : \mathbb{K} \times V \to V$*, satisfying:*
  - *Distributivity over vector addition:* $a \cdot (u + v) = a \cdot u + a \cdot v$*,*
  - *Distributivity over scalar addition:* $(a + b) \cdot u = a \cdot u + b \cdot u$*,*
  - *Associativity:* $(a \cdot b) \cdot u = a \cdot (b \cdot u)$*,*
  - *Multiplicative identity:* $1 \cdot u = u$ *(1 is the identity in* $\mathbb{K}$*).*

Example I.4.

- ***Vector Space on*** $\mathbb{R}^2$*.*
  Consider the set $\mathbb{R}^2$*, which consists of all ordered pairs of real numbers:* $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ *The set* $\mathbb{R}^2$ *is a vector space over the field* $\mathbb{R}$ *with the following operations:*
  - *Vector addition: Given two vectors* $u = (x_1, y_1)$ *and* $v = (x_2, y_2)$*, their sum is defined as:* $u + v = (x_1 + x_2, y_1 + y_2)$
  - *Scalar multiplication: For any scalar* $\alpha \in \mathbb{R}$ *and vector* $u = (x, y)$*, the scalar multiple is:* $\alpha \cdot u = (\alpha x, \alpha y)$
- ***Vector Space of Polynomials.***
  Let $P_n(\mathbb{R})$ *be the set of all polynomials with real coefficients of degree at most* n*. The set*

  $$P_n(\mathbb{R}) = \{p(x) = a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$$

  is a vector space over the field $\mathbb{R}$*. The operations of vector addition and scalar multiplication are defined as follows:*
  - *Vector addition: Given two polynomials* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ *and* $q(x) = b_0 + b_1 x + \cdots + b_n x^n$*, their sum is:*

    $$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

- *Scalar multiplication: For any scalar $\alpha \in \mathbb{R}$ and polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$, the scalar multiplication is:*

$$(\alpha p)(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n$$

- ***Vector Space of Real Sequences***
Consider the set ℓ^2, which consists of all sequences of real numbers that are square-summable, i.e., the set of sequences $(a_1, a_2, a_3, \dots)$ such that

$$\sum_{i=1}^{\infty} a_i^2 < \infty.$$

This set is a vector space over $\mathbb{R}$ with the following operations:

  - *Vector addition: Given two sequences $u = (a_1, a_2, a_3, \dots)$ and $v = (b_1, b_2, b_3, \dots)$, their sum is:*

$$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$$

  - *Scalar multiplication: For any scalar $\alpha \in \mathbb{R}$ and sequence $u = (a_1, a_2, a_3, \dots)$, the scalar multiplication is:*

$$\alpha \cdot u = (\alpha a_1, \alpha a_2, \alpha a_3, \dots)$$

I.3 Normed space

Definition I.4. *Let V denote a linear vector space defined over a field of scalars $\mathbb{R}$ or $\mathbb{C}$. We assume that there exists a real-valued function $\|.\| : E \longrightarrow \mathbb{R}$, that satisfies the following conditions:*

1. $\|x\| \geq 0$, *Positive definiteness.*
2. $\|\lambda x\| = |\lambda|\|x\|$, *Homogeneity.*
3. $\|x + y\| \leq \|x\| + \|y\|$, *Triangle inequality.*
4. $\|x\| = 0$ *if and only if* $x = 0$.

Such a function is called a norm on the vector space V. A vector space equipped with a norm is said to be a normed linear space or simply a normed space.

Example I.5. 1. *The absolute value in $\mathbb{R}$ and the modulus in $\mathbb{C}$ are norms.*

2. ***The euclidean space*** *$\mathbb{R}^n$ equipped with the norm*

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

is a normed space.

3. ***Space of continuous function*** *$(C[a,b])$ endowed with the supremum norm*

$$\|f\|_\infty = \sup |f(x)|$$

is a normed space.

4. ***Sequence space*** *$(l_p)\,, 1 \leq p < +\infty$: The space of infinite sequences $x = (x_1, x_2, \cdots)$ such that certain sums involving the components converge equipped with the norm*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

ia a normed space.

5. *The real function φ defined on $\mathbb{R}^2$ by:*

$$\varphi(x,y) : \longmapsto \varphi(x,y) = |x-y|$$

is not a norm because $\varphi(1,1) = 0$ and $(1,1) \neq (0,0)$.

6. *Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $N : \mathbb{R}^n \to \mathbb{R}$ be defined by:*

$$N(x_1, x_2, \ldots, x_n) = a_1|x_1| + a_2|x_2| + \cdots + a_n|x_n|.$$

Provide a necessary and sufficient condition on a_k for N to be a norm on $\mathbb{R}^n$.

Solution: *The necessary and sufficient condition on a_k for N to satisfy th properties of a norm is*

$$a_k \geq 0 \quad \text{for all } k = 1, 2, \ldots, n, \quad \text{and at least one } a_k > 0.$$

- *The condition $a_k \geq 0$ ensures that $N(x_1, x_2, \ldots, x_n) \geq 0$ for all $(x_1, x_2, \ldots, x_n)$, and $N(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_n = 0$.*
- *The positivity of a_k guarantees the homogeneity and triangle inequality of N.*

Remark I.1. 1. *The closed unit ball in a normed vector space* $(X, \|.\|)$ *is the set of all vectors in E whose norm is less than or equal to 1. More formally, it is expressed as:*

$$B_E = \{x \in X : \ \ \|x\| \leq 1\}.$$

The unit sphere in a normed vector space $(X, \|.\|)$ *is the set of all vectors in E whose norm is exactly 1. Formally, it is defined as*

$$S(0,1) = \{x \in X : \ \ \|x\| = 1\}.$$

2. *In the normed space* E*, we have for all* $x, y \in E$*,*

$$|\|x\| - \|y\|| \leq \|x \mp y\|.$$

Indeed,

$$\|x\| = \|(x \mp y) \pm y\| \leq \|x \mp y\| + \|y\|,$$

so,

$$\|x\| - \|y\| \leq \|x \mp y\|.$$

By interchanging x *and* y*, we get*

$$\|y\| - \|x\| \leq -\|x \pm y\|.$$

I.3.1 Topology associated with a norm

Definition I.5. *Let* $(E, \|\|)$ *be a normed space. The real function defined on* $\mathbb{R}^2$ *by*

$$(x, y) \longmapsto d(x, y) = \|x - y\|$$

constitutes a distance on E*.*

I.3.2 Banach space

This notion of a complete normed space was introduced around 1922 by Stefan Banach.

Definition I.6. *A Banach space is a normed space which is complete i.e., in which every Cauchy sequence is convergent.*
That is, E is complete as a metric space equipped with the distance associated with its norm.

Example I.6. 1. $(\mathbb{R}, |.|)$ *is a Banach space.*

2. *The space of bounded functions $B(E)$, equipped with the norm $\|.\|_\infty$, is a Banach space.*

3. $\big[C([0,1],\mathbb{R}),\|.\|_1\big]$ *is not a Banach space.*
Indeed, let us define in $E=\big[C([0,1],\mathbb{R}),\|.\|_1\big]$ *a sequence* $(f_n)_{n\in\mathbb{N}^*}$ *by:*

$$f_n(x)=\begin{cases} 0 & \text{if } \ 0\le x\le \frac{1}{2} \\ (x-\frac{1}{2})^{\frac{1}{n}} & \text{if } \ \frac{1}{2}\le x\le 1 \end{cases}$$

$(f_n)_{n\in\mathbb{N}^*}$ *is Cauchy sequence.*
Moreover, it is clear that the sequence $(f_n)_{n\in\mathbb{N}^*}$ *converges pointwise to the function* f *defined by:*

$$f(x)=\begin{cases} 0 & \text{if } \ 0\le x\le \frac{1}{2} \\ 1 & \text{if } \ \frac{1}{2}< x\le 1 \end{cases}$$

Furthermore, this sequence converges to f *according to the standard norm* $\|.\|_1$ *since*

$$\begin{aligned} \lim_{n\to\infty}\|f_n-f\|_1 &= \int_{\frac{1}{2}}^{1}(1-(x-\frac{1}{2})^{\frac{1}{n}})dx \\ &= \lim_{n\to\infty}(\frac{1}{2}-\frac{n}{n+1}\frac{1}{2^{\frac{n+1}{n}}})=\frac{1}{2}-\frac{1}{2}=0. \end{aligned}$$

But we find that f *is not continuous at* $x_0=\frac{1}{2}$, *so* E *is not a Banach space.*

4. *Why* $C_0([0,1])$ *is not complete? Let* $C_0([0,1])$ *denote the space of continuous functions* $f:[0,1]\to\mathbb{R}$ *such that* $f(0)=0$, *equipped with the supremum norm:*

$$\|f\|_\infty=\sup_{x\in[0,1]}|f(x)|.$$

Now, consider a different sequence of functions $\{g_n\}_{n=1}^{\infty}$ *defined by:*

$$g_n(x)=x^n.$$

So,

1. *Each* g_n *is continuous on* $[0,1]$.

2. *Each* g_n *satisfies* $g_n(0)=0$, *so* $g_n\in C_0([0,1])$.

The pointwise limit of $\{g_n\}$ *is:*

$$g(x) = \begin{cases} 0 & \textit{if } x \in [0,1[, \\ 1 & \textit{if } x = 1, \end{cases}$$

which is not in $C_0([0,1])$.

Theorem I.1. *Let* E *be a normed space, and let* $(x_n)_{n\in\mathbb{N}^*}$ *be a Cauchy sequence in* E. *Suppose that there exists a subsequence* $(y_p)_{p\in\mathbb{N}^*}$ *of* (x_n) *that converges in* E *to* y, *then* $y = \lim_{n\to\infty} x_n$.

I.3.3 Hilbert Space

A **Hilbert space** is a complete, infinite-dimensional vector space equipped with an inner product. Here are its main properties:

- **Definition** : A Hilbert space $\mathcal{H}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with:
- **Inner product** is a map $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ satisfying:
  - $\langle x, y\rangle = \overline{\langle y, x\rangle}$
  - $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$
  - $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$
- **Completeness** : Every Cauchy sequence converges in $\mathcal{H}$ with respect to the norm induced by the inner product:

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in \mathcal{H}.$$

Example I.7.

- ℓ^2*-space The space of square-summable sequences:*

$$\ell^2 = \left\{ (x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}.$$

- $L^2(\Omega)$*-space: The space of square-integrable functions:*

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \textit{ mesurable} \;\middle|\; \int_\Omega |f|^2 < \infty \right\}.$$

Properties:

- **Orthogonality**: Two elements $x, y \in \mathcal{H}$ are orthogonal if $\langle x, y\rangle = 0$.
- **Orthonormal Basis** : A set $\{e_i\}$ is an orthonormal basis if:

$$\langle e_i, e_j\rangle = \delta_{ij} \quad (\delta_{ij} = 1 \text{ if } i = j,\ 0 \text{ otherwise}).$$

  Every $x \in \mathcal{H}$ admits an expansion $x = \sum_i \langle x, e_i\rangle e_i$.
- **Projection Theorem**: Any vector in $\mathcal{H}$ can be uniquely decomposed as a sum of a vector in a closed subspace and a vector orthogonal to it.

Proposition I.2. *The following chain of strict inclusions holds among complete metric spaces:*

$$\textit{Hilbert spaces} \subsetneq \textit{Banach spaces} \subsetneq \textit{Complete metric spaces.}$$

I.3.4 Finite dimensional normed space

Proposition I.3. *Let E be a finite dimensional normed space, then*

1. *A finite dimensional normed space is isomorphic with $\mathbb{K}^N$ where $N = dimE$.*
2. *In a finite dimensional normed space, a subset is compact if and only if it is closed and bounded.*
3. *A finite dimensional normed space is a Banach space.*
4. *A finite dimensional normed space is closed in that space.*

Exercise: Show the previous properties.

Theorem I.2. *Let $(E, \|.\|)$ be a normed vector space. Then, the closed unit ball is compact if and only if E is finite-dimensional.*

Proof. See [9]. □

I.4 Exercises

Exercise 1 Let $(X, \mathcal{O})$ be a topological space, and let $A \subseteq X$. Show that:

$$\mathrm{int}_X(X \setminus A) = X \setminus \mathrm{adh}_X(A) \quad \text{and} \quad \mathrm{adh}_X(X \setminus A) = X \setminus \mathrm{int}_X(A).$$

Exercise 2 (Continuity and Discrete Topology) Let X be a non-empty set equipped with the discrete topology $\mathcal{P}(X)$, and let $(Y, \mathcal{O}_Y)$ be a topological space. Show that any function $f : X \to Y$ is $(\mathcal{P}(X), \mathcal{O}_Y)$-continuous on X.

Exercise 3 Show that B_0 is a closed subspace of $L^2([0,1])$, where

$$B_0 = \{f \in L^2([0,1]) : |f| \leq 1 \text{ a. e.}\}.$$

Exercise 4 Let $\mathcal{C}^1([0,1])$ be the real vector space of functions $f : [0,1] \to \mathbb{R}$ that are continuously differentiable. We define, for $f \in \mathcal{C}^1([0,1])$, the norm:

$$\|f\| = \left([f(0)]^2 + \int_0^1 |f'(t)|^2 \, dt\right)^{1/2}.$$

1. Prove that this defines a norm on $\mathcal{C}^1([0,1])$.
2. Prove that if the sequence $(f_n)_{n\geq 1}$ converges in this norm, then it converges uniformly on $[0,1]$.
3. Let $f_n(t) = t^n(1-t)$ for $n \geq 1$. Compute $\|f_n\|$.

I.5 Solved problem

Let $\alpha \in]0, 1]$. Define

$$T = \{(x, y) \in [0, 1] \times [0, 1] \ ; \ x \neq y\}.$$

For any function $f : [0, 1] \longrightarrow \mathbb{C}$, define:

$$p_\alpha(f) = |f(0)| + \sup_{(x,y)\in T} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{I.5.1}$$

Let $Lip_\alpha = \{f : [0, 1] \longrightarrow \mathbb{C} \ ; \ P_\alpha(f) < \infty\}$.

1. Prove that $Lip_\alpha \subset C[0, 1]$.
2. Prove that p_α is a norm on Lip_α and that it satisfies, for any $f \in Lip_\alpha$,
$$\|f\|_\infty \leq P_\alpha(f).$$
3. Prove that Lip_α is complete with respect to this norm.

Solution

(1) For a fixed function f, we define $k = p_\alpha(f)$. The definition of $p_\alpha(f)$ leads to the inequality

$$|f(x) - f(y)| \leq k|x - y|^\alpha \quad \text{for all } x, y.$$

This implies the continuity of f.

(2) • Norm value. We need to verify the validity of the norm axioms:

(a) For all $f \in \mathrm{Lip}_\alpha$, $p_\alpha(f) \in \mathbb{R}^+$; and for all $\lambda \in \mathbb{C}$, $p_\alpha(\lambda f) = |\lambda| p_\alpha(f)$. This is easy to verify.

(b) $p_\alpha(f) = 0 \iff f = 0$.

$$p_\alpha(f) = 0 = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} : (x, y) \in J\right\}$$

means that $f(0) = 0$ and $\forall (x, y) \in J, |f(x) - f(y)| = 0$

which implies $\forall (x, y) \in J, f(x) = f(y) = f(0) = 0$

and therefore $f = 0$.

(c) $p_\alpha(f+g) \le p_\alpha(f) + p_\alpha(g)$.

$$\begin{aligned} p_\alpha(f+g) \le |f(0)| &+ |g(0)| \\ &+ \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : (x,y) \in J\right\} \\ &+ \sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\alpha} : (x,y) \in J\right\} \\ &= p_\alpha(f) + p_\alpha(g). \end{aligned}$$

- $\|f\|_\infty \le p_\alpha(f)$. For all $f \in \mathrm{Lip}_\alpha$ and $x \in [0,1]$,

$$\begin{aligned} |f(x)| &\le |f(0)| + |f(x) - f(0)| \\ &\le |f(0)| + (p_\alpha(f) - |f(0)|)|x|^\alpha \\ &\le |f(0)| + (p_\alpha(f) - |f(0)|) = p_\alpha(f). \end{aligned}$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|f\|_\infty \le p_\alpha(f).$$

(3) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Lip_α, and $\epsilon > 0$. There exists $N(\epsilon)$ such that for $p, q \ge N(\epsilon)$, $p_\alpha(f_p - f_q) \le \epsilon$. Since $\|f_p - f_q\|_\infty \le p_\alpha(f_p - f_q)$, the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $(C([0,1]), \|\cdot\|_\infty)$, and therefore converges uniformly to a continuous function f. It remains to show that $f \in \mathrm{Lip}_\alpha$ and that $\lim_{n\to\infty} p_\alpha(f_n - f) = 0$.

Let $p, q \ge N(\epsilon)$, and $(x,y) \in J$. From (I.5.1), we have

$$|f_p(0) - f_q(0)| + \frac{|f_p(x) - f_p(y) - f_q(x) + f_q(y)|}{|x-y|^\alpha} \le p_\alpha(f_p - f_q) \le \epsilon.$$

As $p \ge N(\epsilon)$ is fixed and $q \to \infty$, we obtain

$$|f_p(0) - f(0)| + \frac{|f_p(x) - f(x) - f_p(y) + f(y)|}{|x-y|^\alpha} \le \epsilon.$$

Taking the supremum over $(x,y) \in J$, we see that $f_p - f \in \mathrm{Lip}_\alpha$, which is a vector space, so $f \in \mathrm{Lip}_\alpha$. The inequality obtained can then be interpreted as

$$p \ge N(\epsilon) \implies p_\alpha(f_p - f) \le \epsilon.$$

This completes the proof of the completeness of Lip_α.

Chapter II

Theory of Linear Operators

Contents

II.1 Linear Operator

Definition II.1. *A linear operator T is a function that maps elements between two vector spaces V and W (which can be finite or infinite-dimensional). It satisfies two key properties for all vectors, $x \in V$, $y \in W$ and for any scalar α:*

1. *$T(x+y) = T(x) + T(y)$ (additivity).*
2. *$T(\alpha x) = \alpha T(x)$ (homogeneity).*

Example II.1.

1. ***Differentiation Operator*** *The differentiation operator D, defined on the space of differentiable functions, is linear due to the following property*

$$D(\alpha f) + D(\beta g) = \alpha D(f) + \beta D(g),$$

where f and g are functions, and α and β are scalars.

2. ***Fourier Transform*** *The Fourier transform $\mathcal{F}$ is a linear operator, which transforms a function $f(x)$ into its frequency domain representation. It satisfies the property:*

$$\mathcal{F}(\alpha f) + \mathcal{F}(\beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g),$$

where f and g are functions, and α and β are scalars.

3. ***Integral Operator*** *The integral operator, defined from $C[0,1]$ to $\mathbb{R}$, by:*

$$T(f) = \int_0^1 f(x)dx$$

is a linear operator.

Notation
We will denote the set of linear applications of V in W by $L(V,W)$. We just write $L(V)$ if $V = W$.
The set $L(V,W)$ is endowed with a structure of a linear vector space if we define $(T_1 + T_2)(u) = T_1(u) + T_2(u)$ $u \in V$ and $(\alpha T)(u) = \alpha T(u)$ for all $\alpha \in \mathbb{K}$.

Remark II.1.

1. *If $W = \mathbb{K}$, T is said to be a linear form on V.*
2. *The set of linear forms defined on E is denoted by $E^* = L(E, \mathbb{K})$ and we call it **Algebraic dual** of E.*

3. *E^* has a $\mathbb{K}-$ vector space structure.*

Definition II.2. *Let V, W be two vector spaces on $\mathbb{K}$ and $T : V \to W$. The kernel of the linear operator A is the vector subspace, denoted $KerT \subset E$, defined by:*

$$KerT = \left\{x \in V;\ T(x) = 0_V\right\} = T^{-1}(\{0\}).$$

The image of T, denoted $R(T) \subset W$ or ImT, is defined by:

$$R(T) = \{y = T(x),\ \ x \in V\}.$$

Properties

1. T is injective if and only if $KerT = \{0_V\}$.
2. T is surjective if and only if $R(T) = W$.

Exercice
Show these properties.

II.2 Bounded Linear Operators

Definition II.3. *An operator $T : V \to W$ is called a* ***bounded operator*** *if it maps bounded sets in V into bounded sets in W.*

The following theorem provides a more convenient description of the boundedness of linear operators.

Theorem II.1. *[4] Let V and W be normed spaces and let $T : V \to W$ be a linear operator. T is bounded if and only if there exists a constant K such that*

$$\forall x \in V; \quad \|T(x)\|_W \leq K\|x\|_V.$$

Example II.2. 1. *Consider the linear functional $\varphi : c_0 \to \mathbb{R}$ defined by*

$$\varphi(u) = \sum_{n=1}^{\infty} \frac{u_n}{2^n},$$

where c_0 is the space of sequences that converge to zero.

(a). Show that this functional is continuous and calculate its norm.

(b). Show that $\sup |\varphi(x)|$ is not attained.

Indeed, Since $|u_n|_\infty \leq \|u\|_\infty$ *for all* $n \geq 1$*, we have*

$$|\varphi(u)| \leq \sum_{n=1}^{\infty} \frac{\|u\|_\infty}{2^n} = \|u\|_\infty.$$

Thus, φ *is continuous and* $\|\varphi\| \leq 1$.

To show that $\|\varphi\| = 1$*, consider the element* u_k *in* c_0 *defined by*

$$(u_k)_n = \begin{cases} 1 & \text{if } 1 \leq n \leq k, \\ 0 & \text{if } n \geq k+1. \end{cases}$$

We have $\|u_k\|_\infty = 1$ *and*

$$\varphi(u_k) = \sum_{n=1}^{k} \frac{1}{2^n} = 1 - \frac{1}{2^k}.$$

Thus, as $k \to \infty$*, we have* $\varphi(u_k) \to 1$*, and consequently*

$$\sup_{k \geq 1} |\varphi(u_k)| \geq 1.$$

Therefore, $\|\varphi\| = 1$.

The norm is not attained because for any $u \in c_0$ *with* $\|u\|_\infty = 1$*, there exists an integer* N *such that for all* $n \geq N+1$*, we have* $|u_n| \leq \frac{1}{2}$*. Thus, for large* N*,*

$$|\varphi(u)| \leq \sum_{n=1}^{N} \frac{1}{2^n} + \sum_{n=1}^{N+1} \frac{\frac{1}{2}}{2^n} = \left(1 - \frac{1}{2^N}\right) + \frac{\frac{1}{2}}{2^N} < 1.$$

Hence, the supremum is approached but never attained.

2. *Let* $C^1[0,1]$ *denote the class of all real-valued continuous functions that are continuously differentiable on* $[0,1]$*. Then* $C^1[0,1]$ *is a subspace of* $C[0,1]$*, which is a Banach space with respect to the supremum norm.*

Consider the differential operator

$$T : C^1[0,1] \to C[0,1] \quad \text{defined by} \quad T(f) = f', \quad \text{where } f \in C^1[0,1].$$

We can easily verify that T *is a linear operator, but* T *is not bounded. To illustrate this, let us consider the sequence of functions* $\{f_n\} \subset C^{(1)}[0,1]$ *defined by:*

$$f_n(t) = \sin(n\pi t), \quad 0 \leq t \leq 1.$$

Then, the derivative is given by:

$$T(f_n) = f_n'(t) = n\pi \cos(n\pi t), \quad 0 \leq t \leq 1.$$

- ***Supremum norm of*** f_n***:***

$$\|f_n\| = \sup_{0\le t\le 1} |f_n(t)| = \sup_{0\le t\le 1} |\sin(n\pi t)| = 1.$$

- ***Supremum norm of*** $T(f_n)$***:***

$$\|T(f_n)\| = \|f_n'\| = \sup_{0\le t\le 1} |n\pi\cos(n\pi t)| = n\pi.$$

3. *Let H be a complex, separable, infinite-dimensional Hilbert space. Let (e_n) be an orthonormal basis of H and (λ_n) a sequence of non-zero complex numbers. We define two operators A and B on H by*

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n \quad \textit{and} \quad Bx = \sum_{n=1}^{\infty} \lambda_n^{-1} \langle x, e_n\rangle e_n, \quad \textit{for all } x \in H.$$

   1. *Show that $A \in \mathcal{L}(H)$ if and only if (λ_n) is bounded.*
   2. *Assume that (λ_n) is bounded and* $\inf_n |\lambda_n| > 0$.
      (a) *Find $\|A\|$.*
      (b) *Show that $B \in \mathcal{L}(H)$ and compute Ae_n and Be_n for all $n \in \mathbb{N}$.*

Solution

1. *Proof: $A \in \mathcal{L}(H)$ if and only if (λ_n) is bounded.*
   Recall
   (a) *$\mathcal{L}(H) := \mathcal{L}(H,H)$ is the space of bounded linear operators from H to H.*
   (b) *(e_n) is an orthonormal basis of H, which means*
      i. *(e_n) is orthonormal: For all n and k, $\langle e_n, e_n\rangle = 1$ and $\langle e_n, e_k\rangle = 0$ for $k \ne n$.*
      ii. *(e_n) is total (see the definition).*
      i. *Assume $A \in \mathcal{L}(H)$ and show that (λ_n) is bounded:*
      Let $n \ge 1$ (fixed). Since $Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle e_k$, then

$$\begin{aligned}\|Ae_n\| &= \left\|\sum_{k=1}^{\infty} \lambda_k \langle e_n, e_k\rangle e_k\right\| \\ &= \left\|\lambda_n \langle e_n, e_n\rangle e_n + \sum_{k=1, k\ne n}^{\infty} \lambda_k \langle e_n, e_k\rangle e_k\right\|.\end{aligned}$$

Using $\langle e_n, e_n \rangle = 1$ and $\langle e_n, e_k \rangle = 0$ for all $k \neq n$, we get

$$\|Ae_n\| = \|\lambda_n e_n\| = |\lambda_n|\|e_n\| = |\lambda_n|,$$

since $\|e_n\| = \sqrt{\langle e_n, e_n \rangle} = 1$.
Moreover, as $A \in \mathcal{L}(H)$, we have $\|Ae_n\| \leq \|A\|\|e_n\| = \|A\|$. Thus, $|\lambda_n| = \|Ae_n\| \leq \|A\|$. This shows that (λ_n) is bounded by $\|A\|$.

ii. *Assume (λ_n) is bounded and show that $A \in \mathcal{L}(H)$:*

(a) *Since (λ_n) is bounded, let $M = \sup_n |\lambda_n| < \infty$. Recall that if (e_n) is an orthonormal sequence in a Hilbert space H, then the series $\sum_{n=1}^{\infty} c_n e_n$ converges if and only if $\sum_{n=1}^{\infty} |c_n|^2$ converges. Furthermore, we have*

$$\left\| \sum_{n=1}^{\infty} c_n e_n \right\|^2 = \sum_{n=1}^{\infty} |c_n|^2. \tag{1*}$$

(b) *A is well-defined: We need to show that $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$ converges for $x \in H$. That is, the series $\sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle$ is convergent. For this, it suffices to show that $\sum_{n=1}^{\infty} |\lambda_n \langle x, e_n \rangle|^2$ converges. For all k, we have*

$$\sum_{n=1}^{k} |\lambda_n \langle x, e_n \rangle|^2 = \sum_{n=1}^{k} |\lambda_n|^2 |\langle x, e_n \rangle|^2 \leq M^2 \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2,$$

where (e_n) is an orthonormal basis of H. By Parseval's identity, $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2$. Thus,

$$\sum_{n=1}^{k} |\lambda_n \langle x, e_n \rangle|^2 \leq M^2 \|x\|^2. \tag{2*}$$

This implies that the sequence $\left(\sum_{n=1}^{k} |\lambda_n \langle x, e_n \rangle|^2 \right)_k$ is bounded and hence convergent.

(c) *A is linear and bounded: The linearity of A is immediate* *from its definition. To show boundedness, note that,*

$$\begin{aligned}\|Ax\|^2 &= \left\|\sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n\right\|^2 \\ &= \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, e_n\rangle|^2 \\ &\leq M^2 \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \\ &= M^2\|x\|^2, \textit{ for any } x \in H.\end{aligned}$$

Thus, $\|A\| \leq M < \infty$, so $A \in \mathcal{L}(H)$.

The Bounded Linear Operator Theorem establishes a relationship between a linear operator's boundedness and continuity. It says:

Theorem II.2. *[4] Let $(E, \|.\|)$, $(F, \|.\|)$ and be two normed spaces, and let $T \in L(V, W)$. T is continuous if and only if it is bounded.*

Proof. 1. The condition is necessary: Since T is continuous, in particular at zero,

$$\forall \epsilon > 0, \;\; \exists \rho > 0 \;\; \|x\| < \rho \Rightarrow \|T(x)\| < \epsilon.$$

Let us set $\epsilon = 1$ and let us define, for all $x \neq 0$, $z = \dfrac{\rho}{2\|x\|}x$.

We have $\|z\| = \dfrac{\rho}{2} < \rho$, so,

$$T\left(\frac{\rho}{2\|x\|}x\right) < 1 \Rightarrow \frac{\rho}{2\|x\|}\|T(x)\| < 1 \Rightarrow \|T(x)\| < \frac{2}{\rho}\|x\|.$$

Hence, T is bounded.

2. The condition is sufficient. For all $x, y \in E$, we obtain

$$\|T(x) - T(y)\| = \|T(x-y)\| \leq C\|x-y\|.$$

Therefore, T is Lipschitz continuous.

□

Below are key properties of bounded linear operators:

Lemma II.1. *[4] Let $T \in L(V, W)$. The following assertions are equivalent:*

1. *T is continuous on V.*
2. *T is continuous at zero.*
3. *T is uniformly continuous.*

Proof.

1. $1 \Rightarrow 2$: Obvious.
2. $2 \Rightarrow 3$: The fact that T is continuous at zero implies that
$$\forall x \in E; \quad \|A(x)\| \leq C\|x\|.$$
This yields
$$\forall x, y \in E; \quad \|A(x) - A(y)\| \leq \|A(x - y)\| \leq C\|x - y\|,$$
Thus, T is uniformly continuous..
3. $3 \Rightarrow 1$: Trivial.

$\square$

Notations

1. The set of bounded linear operators from V to W is a vector subspace of $L(E, F)$, and is denoted by $\mathcal{L}(E, F)$.
2. Similarly, the linear functionals on V form a subspace of the algebraic dual space $V^\star$. This subspace is denoted by V' and is called the topological dual of E.
3. Finally, note that if $V = W$, then $\mathcal{L}(V, W) = \mathcal{L}(E)$.

Theorem II.3. *The map $\|.\| : \mathcal{L}(E, F) \to \mathbb{R}$ defined by*
$$\begin{aligned}\|A\| &= \sup\left\{\|A(x)\|;\ x \in E \text{ and } \|x\| \leq 1\right\}\\ &= \sup_{\|x\|\leq 1} \|A(x)\|\end{aligned}$$
is a norm on $\mathcal{L}(E, F)$

Proof. 1. For $A \in \mathcal{L}(E,F)$, $\|A\| \geq 0$.
If $\|A\| = 0$, then $Ax = 0$, for all $x \in E$. Indeed, for $x \neq 0$,
$\|A(x)\| = \|x\| \left\| A(\frac{x}{\|x\|}) \right\|$, so $A(x) = 0$ on E, hence $A = 0$.

2. For all $\lambda \in \mathbb{K}$, $\|\lambda A\| = |\lambda| \|A\|$, since

$$\|\lambda A\| = \sup_{\|x\| \leq 1} \|\lambda A(x)\| = |\lambda| \sup_{\|x\| \leq 1} \|A(x)\| = |\lambda| \|A\|.$$

3. For all $x \in E$, such that $\|x\| \leq 1$, it holds that

$$\begin{aligned} \|(A+B)(x)\| &= \|A(x) + B(x)\| \\ &\leq \|A(x)\| + \|B(x)\| \\ &\leq \|A\| + \|B\|. \end{aligned}$$

Therefore, $\|A + B\| \leq \|A\| + \|B\|$.

□

Proposition II.1. *Let T be a bounded linear operator. Then, we have*

$$\begin{aligned} \|A\| &= \sup_{\|x\| \leq 1} \|A(x)\| = \sup_{\|x\| = 1} \|A(x)\| = \sup_{x \neq 0} \frac{\|A(x)\|}{\|x\|} \\ &= \inf \left\{ M;\ \ \|A(x)\| \leq M\|x\|, \ \textit{pour tout} \ \ x \in E \right\}. \end{aligned}$$

Proof. See [4]. □

Example II.3. *Let*

$$\begin{aligned} T : \Big((C[0,1], \mathbb{R}), \|.\|_\infty \Big) &\to \Big((C[0,1], \mathbb{R}), \|.\|_\infty \Big) \\ f &\to T(f)(x) = \int_0^x f(t)dt. \end{aligned}$$

- *Determine the norm of T.*

Solution

- *Let's show that $\|T\| = 1$.*

- $\forall f \in \Big(\big(C[0,1], \mathbb{R}\big); \|.\|_\infty\Big)$,

$$\begin{aligned} \|T(f)\|_\infty &= \sup_{[0,1]} |T(f)(x)| = \sup_{[0,1]} \left| \int_0^x f(t)dt \right| \\ &\leq \int_0^1 |f(t)|dt \leq \sup_{0\leq t\leq 1} |f(t)| \int_0^1 dt = \|f\|_\infty. \end{aligned}$$

Hence, $\|T\| \leq 1$.

- *By taking* $f = 1$, *then* $f \in \Big(\big(C[0,1], \mathbb{R}\big), \|.\|_\infty\Big)$ *and we get*

$$\begin{aligned} \|T(f)\|_\infty &= \sup_{0\leq x\leq 1} \left| \int_0^x f(t)dt \right| \\ &= \sup_{0\leq x\leq 1} |x| = 1. \end{aligned}$$

As a result $\|T\| \geq 1$. *In conclusion,* $\|T\| = 1$.

Theorem II.4. *[6] Let* $(E, \|.\|_1)$ *be a be a* ***normed*** *space and* $(F, \|.\|_2)$ *a* ***Banach*** *space. Then* $\mathcal{L}(E, F)$ *is a Banach space.*

Proof. It is already known that $\mathcal{L}(E, F)$ is a normed space. It's remains to prove that $\mathcal{L}(E, F)$ is complete.
Let $\big\{A_n, n \in \mathbb{N}\big\} \subset \mathcal{L}(E, F)$ be a Cauchy sequence. Then,

$$\|A_n - A_m\| \leq \epsilon, \text{ if } m, n \geq N_0(\epsilon).$$

For a fixed x, we have

$$\|A_n(x) - A_m(x)\|_2 \leq \|A_n - A_m\|\|x\|_1 \leq \epsilon\|x\|_1, \quad n, m \geq N_0(\epsilon).$$

This shows that $\big\{A_n(x), n \in \mathbb{N}\big\} \subset F$ is a Cauchy sequence for all $x \in E$. Now, $(F, \|.\|_2)$ is a Banach space, so $\big\{A_n(x)\big\}_n$ is convergent for all $x \in E$. We define the operator $A : E \to F$ as follows

$$A(x) = \lim_n A_n(x), \quad \text{for all } x \in E.$$

Let's show that for all $x, y \in E$,$\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} A(\alpha x + \beta y) &= \lim_n A_n(\alpha x + \beta y) = \alpha \lim_n A_n(x) + \beta \lim_n A_n(y) \\ &= \alpha A(x) + \beta A(y), \end{aligned}$$

On the other hand, we have

$$\|A_n(x)\|_2 \leq M\|x\|_1 \quad \forall x \in E,$$

This leads, as $n \to \infty$, to $\|A(x)\|_2 \leq M\|x\|_1, \quad \forall x \in E,$ and thus $A \in \mathcal{L}(E, F)$.
Finally, let's show that $A_n \xrightarrow{\|\cdot\|} A$.
If $m, n \geq N_0(\epsilon),\ x \in E$, then

$$\|A_m(x) - A_n(x)\| \leq \|A_m - A_n\|\|x\|_1 \leq \epsilon\|x\|_1, \quad \text{if } m, n \geq N_0(\epsilon).$$

This gives

$$\|A - A_n\| < \epsilon, \quad \text{if } n \geq N_0(\epsilon), \text{ i.e } A_n \xrightarrow{\|\cdot\|} A.$$

□

Corollary II.1. *Let $(E, \|\|)$ be a normed space. Then the topological dual space E' is a Banach space.*

Proof. The completeness of the space E' comes from the fact that $\mathbb{K}$ is complete. □

II.3 Extension by continuity

We remind the lecture of the result presented by the following theorem.
Reminder

Theorem II.5. *[9] Let (E, d) and (F, ρ) be two metric spaces with F complete and X be a dense subspace of E.*
If $f : X \to F$ is a uniformly continuous operator, then f can be uniquely extended to a uniformly continuous operator from $E \to F$.

Proof. Let $f : X \to F$ be uniformly continuous.
If $x \in \overline{X}$, there exists $\{a_n\}_n \subset X$ such that $\lim_n a_n = x$.
Thus, $\{a_n\}_n$ is a Cauchy sequence in X and since f is uniformly continuous, then $\{f(a_n)\}_n \subset F$ is a Cauchy sequence, which implies the existence of

$$y \in F \text{ such that } \lim_n f(a_n) = y.$$

We define

$$g(x) = y = \lim_n f(a_n).$$

- g is well-defined.
  Indeed, Suppose that $a_n \underset{n}{\longrightarrow} x$ and $b_n \underset{n}{\longrightarrow} x$.
  Let's show that
$$\lim_n f(a_n) = \lim_n f(b_n) = y.$$
$$d(a_n, b_n) \leq d(a_n, x) + d(b_n, x) \to 0, \text{ when } n \to \infty.$$
  And since f is uniformly continuous, we have
$$\rho(f(a_n), f(b_n)) \to 0 \text{ when } n \to \infty.$$
  Which gives
$$\rho(f(b_n), y) \leq \rho(f(a_n), f(b_n)) + \rho(f(a_n), y) \to 0, \text{ when } n \to \infty,$$
  thus $\lim_n f(b_n) = y$, and g is well-defined.

- Now, we show that g is uniformly continuous on E.
  Since f is uniformly continuous on X.
$$\forall \epsilon > 0, \ \exists \delta > 0, \ \forall x_1, x_2 \in X; d(x_1, x_2) < \delta \Rightarrow \rho(f(x_1), f(x_2)) < \epsilon.$$
  Let $x, y \in E$, then there exist sequences $(a_n)_n, (b_n)_n \subset X$, such that $\lim_n a_n = x$ and $\lim_n b_n = y$.
  If $d(x, y) < \delta$, then $d(a_n, b_n) \leq d(a_n, x) + d(x, y) + d(y, b_n) \leq \delta$, for sufficiently large n .
  Which implies that, for sufficiently large n, $\rho(f(a_n), f(b_n)) < \epsilon$ Thus,
$$\rho(g(x), g(y)) = \lim_n \rho(f(a_n), f(b_n)) < \epsilon.$$
  Therefore, g is uniformly continuous.
  We assume that there exists another extension h, such that
$$h|_A = f = g.$$
$$\text{for } x \in \overline{X}, \text{ there exists } (a_n) \subset X \text{ such that } a_n \underset{n}{\longrightarrow} x.$$
  Thus,
$$g(x) = \lim_n g(a_n) = \lim_n h(a_n) = h(x)$$

$\square$

Now, let's present a result that goes in the direction of the extension theorem mentioned below.

Lemma II.2. *[9] let E_0 be a subspace of a normed space E, F a Banach space, and $A_0 : E_0 \to F$ a continuous linear operator. Then, there exists a unique continuous linear extension A of A_0 on $\overline{E_0}$ such that $\|A\| = \|A_0\|$.*

Proof. A_0 is uniformly continuous because A_0 is linear and continuous. Therefore, by Theorem (II.5), there exists a unique uniformly continuous extension A of A_0 on $\overline{E_0}$.

- Let's show that A is linear.
  Since $A(x) = \lim_n A_0(a_n)$, one has

$$\begin{aligned} A(\alpha x + \beta y) &= \lim_n A_0(\alpha a_n + \beta b_n) \\ &= \alpha \lim_n A_0(\alpha a_n) + \beta \lim_n A_0(\alpha b_n) \\ &= \alpha A(x) + \beta A(y). \end{aligned}$$

  Hence, A is linear.

- Now, let's prove that A is bounded.
  Since

$$\|A(x)\| = \lim_n \|A_0(a_n)\| \leq \|A_0\| \lim_n \|a_n\| \leq \|A_0\|\|x\|.$$

  Thus, A is bounded and

$$\|A\| \leq \|A_0\|. \tag{II.3.1}$$

  On the other hand,

$$\begin{aligned} \|A_0\| &= \sup_{\substack{\|x\|\leq 1 \\ x\in E_0}} \|A_0(x)\| = \sup_{\substack{\|x\|\leq 1 \\ x\in E_0}} \|A(x)\| \\ &\leq \sup_{\substack{\|x\|\leq 1 \\ x\in \overline{E_0}}} \|A(x)\| = \|A\|. \end{aligned} \tag{II.3.2}$$

  Form (II.3.1) and (II.3.2), we obtain

$$\|A\| = \|A_0\|.$$

□

The following result specifies the relationship between the finite dimension of the space and the continuity of the operator.

Theorem II.6. *Let E, F be two normed spaces and $A : E \to F$ a linear operator. If $dimE < \infty$, then A is continuous.*

Proof. Let E be a finite-dimensional vector space and $\{e_1, e_2, \ldots, e_n\}$ a basis of E. Then, for all $x \in E,\ x = \sum_{i=1}^{n} x_i e_i,\ x_i \in \mathbb{K}$ and $i = 1, 2, \ldots, n$,

$$\begin{aligned} \|A(x)\| &\leq \sum_{i=1}^{n} |x_i| \|A(e_i)\| \leq \max_{0<i<n} \|A(e_i)\| \sum_{i=1}^{n} |x_i| \\ &= \max_{0<i<n} \|A(e_i)\| \|x\|_\infty \leq K\|x\|_\infty. \end{aligned}$$

□

II.4 Pointwise convergence and uniform convergence

II.4.1 Pointwise convergence

Definition II.4. *A sequence of functions $(f_n)_n$ defined on a set E converges pointwise to a function f iff, for each $x \in E$, the sequence $(f_n(x))_n$ converges to $f(x)$. In other words, for every $x \in E$, we have*

$$\lim_n f_n(x)dx = f(x).$$

This means that for each point $x \in E$,

$$f_n(x) \xrightarrow{\text{Pointwise}} f(x) \iff \forall x \in E,\ \lim_n f_n(x)dx = f(x).$$

Example II.4. *The sequence of functions $f_n(x) = x^n$ converges pointwise to the function*

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

II.4.2 Uniform convergence

Definition II.5. *A sequence of functions $\{f_n\}$ defined on a set E converges uniformly to a function f if, for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$ and for all $x \in E$, we have:*

$$|f_n(x) - f(x)| < \epsilon.$$

That is:

$$f_n \xrightarrow{\text{uniformly}} f \iff \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in E, \quad |f_n(x) - f(x)| < \epsilon.$$

Example II.5. *Study the uniform convergence of the sequence of functions, defined on $[0,1]$, by:*

$$f_n(x) = \frac{ne^{-x} + x^2}{n + x}.$$

1. *Pointwise convergence: $f_n(x) \to f(x) = e^{-x}$, because*

$$\lim_n f_n(x) = \lim_n \frac{ne^{-x}}{n} = e^{-x}.$$

2. *Uniform convergence:*

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{ne^{-x} + x^2}{n + x} - e^{-x} \right| = \left| \frac{ne^{-x} + x^2 - ne^{-x} - xe^{-x}}{n + x} \right| \\ &= \left| \frac{x(x - e^{-x})}{n + x} \right| \leq \frac{|x| + e^{-x}}{n + x} \leq \frac{2}{n} \xrightarrow[n]{} 0. \end{aligned}$$

Thus, f_n converges uniformly to e^{-x} on $[0,1]$.

As a consequence of uniform convergence, we obtain the following main results.

Uniform Convergence and Continuity

Proposition II.2. *If $\{f_n\}$ converges uniformly to f, this means that for any $\epsilon > 0$, there exists an integer N such that for all $n \geq N$ and for all $x \in E$, we have*

$$|f_n(x) - f(x)| < \epsilon.$$

Now, if each f_n is continuous, and $f_n \to f$ uniformly, then the limit function f must be continuous. This follows because, for any $x_0 \in E$ and any $\epsilon > 0$, we can find an N such that for all $n \geq N$, we have

$$|f_n(x_0) - f(x_0)| < \epsilon.$$

By the continuity of each f_n, for x near x_0, we have

$$|f_n(x) - f_n(x_0)| < \epsilon.$$

Therefore, the uniform convergence ensures that the continuity of each function f_n passes to the limit function f.

Example II.6. *The sequence of continuous functions defined by $f_n(x) = x^n$ does not converge uniformly on $[0, 1]$, because f_n converges pointwise to f defined by:*

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } 0 \leq x < 1, \end{cases}$$

which is not continuous.

Uniform Convergence and Integration

Proposition II.3 (Integration of the Limit Function)**.** *If f_n converges uniformly to f on $[a, b]$, then the integral of f_n converges to the integral of f over the same interval. That is*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Proposition II.4 (Convergence of Integrals)**.** *Uniform convergence ensures that the integrals of the sequence f_n converge to the integral of the limit function f. This follows from the Dominated Convergence Theorem, which states that if f_n converges uniformly to f, then*

$$\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx \text{ as } n \to \infty.$$

Proposition II.5 (Pointwise Convergence)**.** *If the convergence of f_n to f is only pointwise (not uniform), then the integral of f_n may not converge to the integral of f. In such cases, additional conditions like the Dominated Convergence Theorem or the Monotone Convergence Theorem are needed to guarantee the convergence of the integrals.*

Example II.7. *Let $f_n: [0,1] \longrightarrow \mathbb{R}$ be defined by*

$$f_n(x) = \begin{cases} n^2x(1-nx) & \text{for } x \in \left[0, \frac{1}{n}\right], \\ 0 & \text{otherwise.} \end{cases}$$

1. *Study the pointwise limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$.*
2. *Compute $\displaystyle\int_0^1 f_n(t)\,dt$.*
   Is there a uniform convergence of the sequence $(f_n)_{n\in\mathbb{N}}$?

Solution.

1. For $x = 0$, we have $f_n(x) = 0$.
   For $x \in]0,1]$, there exists N_0 such that $\dfrac{1}{x} < N_0$, so for all $n \geq N_0$, we have $f_n(x) = 0 \longrightarrow 0$.
   Thus, the sequence of functions converges pointwise to 0 on $[0,1]$.
2. $\displaystyle\int_0^1 f_n(t)\,dt = \frac{1}{6}$.
   If there had been uniform convergence of the sequence of functions $(f_n)_{n\in\mathbb{N}}$, we would have $\displaystyle\int_0^1 f_n(t)\,dt = \int_0^1 0\,dx = 0$.
   This is not the case, so there is no uniform convergence.

Uniform Convergence and Differentiability

Theorem II.7. *Let $(f_n)_n$ be a sequence of functions, taking values in $\mathbb{R}$ or $\mathbb{C}$, defined on an interval I.*
We assume that all the f_n are of class C^1 and that there exists $g: I \longrightarrow \mathbb{R}$ satisfying the following conditions:

- *There exists $x_0 \in I$, such that the sequence $(f_n(x_0))_{n\in\mathbb{N}}$ converges.*
- *The sequence of functions f_n' converges uniformly to g on every subinterval contained in I.*

Then, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on I to a function f, continuously differentiable on I, such that $f' = g$.

Example II.8. *Let* $(f_n)_{n\in\mathbb{N}^*}$ *be the sequence of functions defined on* $[-1,1]$ *by:*

$$f_n(x) = \frac{x}{1+n^2x^2}.$$

1. *Show that* $(f_n)_{n\in\mathbb{N}^*}$ *converges uniformly on* $[-1,1]$ *to* 0.

2. *Consider the sequence* g_n *defined on* $[-1,1]$ *by*

$$g_n(x) = \frac{\ln(1+n^2x^2)}{2n^2}.$$

Show that $(g_n)_{n\in\mathbb{N}^*}$ *converges uniformly on* $[-1,1]$ *to* 0.

Solution

$(f_n)_{n\in\mathbb{N}^*}$ *converges pointwise to* 0.

1. *We will study the function*

$$h_n(x) = |f_n - 0| = \frac{|x|}{1+n^2x^2}.$$

Since the function is even, we can restrict the analysis to $[0,1]$*, thus we have*

$$h_n(x) = \frac{x}{1+n^2x^2}$$

and

$$h_n'(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}.$$

It follows that

$$\sup_{0\le x\le 1} |h_n| = \frac{1}{2n} \to 0 \text{ as } n \to \infty.$$

Thus, the sequence converges uniformly to 0 *on* $[-1,1]$.

2. $g_n' = \dfrac{x}{1+n^2x^2} = f_n(x)$

Therefore, g_n' *converges uniformly on* $[-1,1]$ *to* 0.
According to the differentiation theorem, the sequence $(g_n)_{n\in\mathbb{N}^*}$ *converges uniformly to* 0.

II.5 Invertibility of an operator

Definition II.6. *An operator T is said to be invertible if there exists another operator T^{-1} such that*

$$TT^{-1} = T^{-1}T = I,$$

where I is the identity operator.

Definition II.7. *Let E be a Banach space. We recall that an operator $T \in \mathcal{L}(E)$ is invertible if it has an inverse in $\mathcal{L}(E)$, i.e.,*

$$\exists S \in \mathcal{L}(E) \text{ such that } ST = TS = I,$$

where I denotes the identity operator on E.
We denote by $GL(E)$ the set of invertible operators.

Example II.9. *Let $E = l_2(\mathbb{R}) = \{(x_1, x_2, \ldots, x_n, \ldots); \sum_{i=1}^{\infty} x_i^2 < \infty\}$, and define the operator $T : E \longrightarrow F$ by:*

$$T(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots), \quad \forall x = (x_n)_{n\geq 1} \in E.$$

Then, T is injective and $T^{-1} \in \mathcal{L}(T(E))$, but not in $\mathcal{L}(E)$, so T is not invertible.

Theorem II.8. *[1][Open Mapping Theorem] Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear operator. If T is surjective, then*

$$\exists c > 0, \quad \text{such that } B_Y(0, c) \subset T(B_X(0, 1)). \qquad (*)$$

In particular, T is an open map, meaning that for every open set $U \subseteq X$, the image $T(U)$ is an open set in Y.

The (*) property implies that

$$\exists c > 0, \quad \text{such that } B_Y(0, Rc) \subset T(B_X(0, R)). \qquad (**)$$

Indeed,

$$\|y\| \leq r \Rightarrow \|\frac{y}{r}\| \leq 1 \Rightarrow \exists x \in B_X(0, R) : \frac{y}{r} = T(x),$$

which gives

$$y = T(rx) = T(z); \quad z \in B_X(0, R);$$

thus $B_Y(0, Rc) \subset T(B_X(0, R))$.

Proof. **Show that T is open.** Let U be a non-empty open set in X. Let $y_0 \in T(U)$. Then there exists $x_0 \in U$ such that $y_0 = T(x_0)$.
There exists $r > 0$ such that $B_X(0, R) \subset U$. We have

$$\begin{aligned} y_0 + T(B_X(0, R)) &= T(x_0) + T(B_X(0, R)) \\ &= T\left(x_0 + B_X(0, R)\right) \\ &= T\left(B_X(x_0, R)\right) \\ &\subset T(U). \end{aligned}$$

On the other hand, we also have:

$$B_Y(y_0, Rc) = y_0 + B_Y(0, Rc) \subset y_0 + T(B_X(0, R)) \subset T(U).$$

□

Theorem II.9. *[1][Banach isomorphism Theorem] Let E and F be Banach spaces and let $T : E \to F$ be a linear, continuous, and bijective operator. Then, T^{-1} is also continuous (and so T is a topological isomorphism).*

Proof. Since A is bijective, A^{-1} exists.
By Theorem II.8, A is an open map. Thus, for any open set G in E, $A(G)$ is open in F.
Hence, A^{-1} is continuous, since $(A^{-1})^{-1}(G) = A(G)$ is open. □

Proposition II.6. *[7] Let $(X, \|\cdot\|_X)$ be a normed vector space over $\mathbb{K}$, E be a Banach space over the same field $\mathbb{K}$, and $T \in \mathcal{L}(E, X)$ be bijective. Then the following statements are equivalent:*

1. *$T^{-1} \in \mathcal{L}(E, X)$.*
2. *There exists $C > 0$ such that for all $x \in E$, $\|T(x)\| \geq C\|x\|$.*
3. *$(X, \|\cdot\|_X)$ is a Banach space.*

Proof. . Suppose T^{-1} is continuous. Then

$$\forall x \in E,\ \|T^{-1}(T(x))\| \leq \|T^{-1}\|\|T(x)\|.$$

Thus,

$$\forall x \in E,\ \|T(x)\|_X \geq \|T^{-1}\|^{-1}\|x\|_E.$$

Let $C = \|T^{-1}\|^{-1}$, which shows statement 2.

- Let $(y_n)_{n\geq 1}$ be a Cauchy sequence in X. Since the operator T is bijective, there exists a sequence $(x_n)_{n\geq 1} \subset E$ such that, for all $n \in \mathbb{N}$,

$$Tx_n = y_n.$$

Now, by statement 2,

$$\|x_p - x_q\| \leq \frac{1}{C}\|y_p - y_q\|.$$

Thus, $(x_n)_{n\geq 1}$ is a Cauchy sequence in E (Banach), and $x_n \xrightarrow{E} x$. Since T is continuous, we have $Tx_n \xrightarrow{Y} Tx \in X$. Therefore, y_n converges to $Tx \in X$, and we deduce that X is a Banach space.

- The Banach isomorphism Theorem (II.9) completes the proof.

□

Corollary II.2. *Let $(E, \|.\|)$, $(Y, \|.\|)$ be two Banach spaces over $\mathbb{K}$, and let $T \in \mathcal{L}(E, Y)$. Then, the following properties are equivalent*

1. *There exists $C > 0$ such that for all $x \in E$, $\|T(x)\| \geq C\|x\|$.*
2. *T is injective and $\mathrm{Im}(T)$ is closed in Y.*

Proof. We set $X = \mathrm{Im}(T)$.

- T is injective because $Ker(T) = \{0\}$ ($x \in Ker(T)$ implies that $\|T(x)\| = 0 \geq C\|x\|$).
Consequently,
T is a bijection from E onto X. According to Proposition (II.6), X is a Banach space and thus closed in Y.

- Now, if T is injective and X is closed in Y, then T is a bijection from E onto X, which is a Banach space. Proposition (II.6) states that there exists a constant $C > 0$ such that, for all $x \in E$,

$$\|T(x)\| \geq C\|x\|.$$

This proves the Corollary.

□

II.6 Topological dual of a normed vector space.

Definition II.8. *Let E be a normed space. The space $L(E;\mathbb{R})$ of all linear bounded functionals $f : E \to \mathbb{R}$ is called the dual of E and it is denoted by E^*. The norm on E^* is therefore defined by*

$$\|f\|_{E^*} = \sup_{\|x\|\leq 1} |f(x)| = \sup_{\|x\|=1} |f(x)|.$$

Example II.10. *In the space $l_p(\mathbb{R}) = \big\{(x_1, x_2, \ldots, x_n, \ldots);\ \sum_{i=1}^{\infty} |x|_i^p < \infty\big\}$, let $e_n = (0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots) \in l_p$,*

$x = (x_1, x_2, \ldots, x_n, \ldots)$ and the sequence $\xi_n = \sum_{i=1}^{n} x_i e_i = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$.
Then

$$\|x - \xi_n\|_p = \big[\sum_{i=n+1}^{\infty} |x_i|^p\big]^{\frac{1}{p}}$$

and since $\big[\sum_{i=n+1}^{\infty} |x_i|^p\big]$ converges, we have for any $\epsilon > 0$, there exists $N(\epsilon)$ such that $\|x - \xi_n\|_p < \epsilon$ for $n \geq N(\epsilon)$.
Let $f \in l_p'$ and $y_i = f(e_i) \in \mathbb{K}$.
Thus, $|y_i| = |f(e_i)| \leq \|f\|\|e_i\| \leq \|f\|$.
From the relation

$$f(x - \xi_n) = f(x) - \sum_{i=1}^{n} x_i f(e_i) = f(x) - \sum_{i=1}^{n} x_i y_i,$$

we get

$$f(x - \xi_n) \leq \|f\|\|x - \xi_n\| \leq \|f\|\epsilon, \text{for all } n \geq N(\epsilon).$$

Thus, $f(x) = \sum_{i=1}^{n} x_i y_i$.

- *Let us show that $\{y_n\}_n$ is a set in l_q where $q = \dfrac{p}{p-1}$, $p > 1$.*

Let

$$z_i = \begin{cases} 0 & \text{if } y_i = 0, \\ \dfrac{|y_i|^q}{y_i} & \text{if } y_i \neq 0. \end{cases}$$

We define $\zeta_n = \sum_{i=1}^{n} z_i e_i$. *Then, we have*

$$\|\zeta_n\|_p = \Big[\sum_{i=1}^{n} |z_i|^p\Big]^{\frac{1}{p}} = \Big[\sum_{i=1}^{n} |y_i|^{p(q-1)}\Big]^{\frac{1}{p}} = \Big[\sum_{i=1}^{n} |y_i|^q\Big]^{\frac{1}{p}}.$$

Since $f(\zeta_n) = \sum_{i=1}^{n} z_i y_i = \sum_{i=1}^{n} |y_i|^q$, *we obtain*

$$\begin{aligned}\sum_{i=1}^{n} |y_i|^q &= f(\zeta_n) = |f(\zeta_n)| \leq \|f\|\|\zeta_n\|_p \\ &= \|f\|\Big[\sum_{i=1}^{n} |y_i|^q\Big]^{\frac{1}{p}}.\end{aligned}$$

Hence, $\Big[\sum_{i=1}^{n} |y_i|^q\Big]^{\frac{1}{q}} \leq \|f\|$, *and by passing to the limit, we have*

$$\Big[\sum_{i=1}^{\infty} |y_i|^q\Big]^{\frac{1}{q}} \leq \|f\|.$$

In other words, $y = \{y_n\} \in l_q$ *and* $\|f\| \geq \|y\|_q$.
On the other hand, Hölder's inequality gives us

$$\text{for } x \in l_p, \quad f(x) = \sum_{i=1}^{\infty} x_n y_n \leq \|x\|_p \|y\|_q,$$

which implies $\|f\| \leq \|y\|_q$.
In conclusion, $\|f\| = \|y\|_q$.

Example II.11. *Let* ℓ_1 *denote the space of infinite sequences* (x_n) *such that* $\sum_{n=1}^{\infty} |x_n| < \infty$. *We equip it with the norm* $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$. *Let* ℓ_∞ *denote the space of bounded infinite sequences* (x_n). *We equip it with the norm* $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. *We claim that the dual of* ℓ_1 *is isometrically isomorphic to* ℓ_∞.

Solution

The isometric isomorphism is given by

$$T : \ell_\infty \to (\ell_1)^*, \quad a \mapsto T_a : \ell_1 \to \mathbb{R}, \quad x \mapsto \sum_{n=1}^{\infty} a_n x_n.$$

. *The linearity of* T_a *in* x *and the linearity of* T *in* a *are clear.*

- *The continuity of T_a in x follows from the following inequality*

$$\|T_a(x)\| = \sum_{n=1}^{\infty} a_n x_n \leq \sum_{n=1}^{\infty} |a_n x_n| \leq \sum_{n=1}^{\infty} |x_n||a_n|\|a\|_1 \leq \|a\|_1\|x\|_1.$$

- *The surjectivity: let $e_1 = (1,0,0,\ldots)$, $\quad e_2 = (0,1,0,0,\ldots)$, and so on. The set $\{e_1, e_2, e_3, \ldots, \}$ is a Schauder basis of ℓ^1. If $x = (x_1, x_2, \ldots) \in \ell^1$, the series $\sum_{n=1}^{\infty} x_n e_n$ converges in the ℓ^1 norm to x because*

$$\begin{aligned} \|x - \sum_{k=1}^{n} x_k e_k\|_1 &= \|(x_1, \ldots, x_n, x_{n+1}, \ldots) - (x_1, \ldots, x_n, 0, 0, \ldots)\|_1 \\ &= \|(0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)\|_1 \\ &= \sum_{k=n+1}^{\infty} |x_k| \underset{n \to \infty}{\longrightarrow} 0. \end{aligned}$$

If now $f \in (\ell^1)*$, then f is linear and continuous and so

$$f(x) = f\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n f(e_n).$$

Let $a_n = f(e_n)$. Since f is bounded, the sequence (a_n) is bounded and so, $a := (a_1, a_2, \ldots, a_n, \ldots) \in \ell^\infty$. Thus

$$f(x) = \sum_{n=1}^{\infty} a_n x_n = (T_a)(x).$$

It follows that $f = T_a$.

- *The isometry:*

$$\|(Ta)(x)\| \leq \|a\|_{\ell^1}\|x\|_{\ell^1} :$$

It follows that $\|Ta\| \leq \|a\|_{\ell^1}$. On the other hand, by property of supremum, There exists $k \in \mathbb{N}$ such that $|a_k| > \|a\|_{\ell^1} - \epsilon$.
A simple calculation leads to

$$\|Ta\| \leq \|(Ta)(e_k)\| = |a_k| \leq \|a\|_{\ell^1} :$$

Since k was arbitrary, we get $\|Ta\| \leq \|a\|_{\ell^1}$.

$$\|Ta\| \geq \|(Ta)(e_k)\| = |a_k| \geq \|a\|_{\ell^\infty} - \epsilon.$$

Since ϵ was arbitrary, we get $\|Ta\| \geq \|a\|_{\ell^\infty}$. Which completes the proof.

Theorem II.10. *[9][Riesz Representation Theorem] Let F be a linear form on $L^p(E) = \{f : E \longrightarrow \mathbb{R}; f$ is measurable and $\int_E |f|^p dx < \infty\}$, $1 < p < \infty$. Then, there exists a function $g \in L^q(E)$, $q = \frac{p}{p-1}$, such that*

$$F(f) = \int_E f(x)g(x)\, dx, \text{ for all } f \in L^p(E).$$

Moreover, $\|F\| = \|g\|$.

Proof. See [9]. □

Remark II.2. *Let $\tau : L_q(E) \longrightarrow (L_p(E))'$ be defined by:*

$$\langle \tau(g), f\rangle = F(f) = \int_E f(x)g(x)\, dx.$$

Then, τ is an isomorphism and isometric (linear, bijective, and an isometry).

II.7 Bidual. Reflexive space.

Definition II.9. *Let E be a normed space and E' its dual, which is also a Banach space. Then, $(E')'$, also denoted as E'', which is the space of continuous linear functionals on E', is a Banach space known as the bidual of E. The norm of F in E'' is defined as follows:*

$$\|F\|_{E''} = \sup_{\|f\|_{E'} \le 1} |F(f)| = \sup_{f \neq 0,\, f \in E'} \frac{|F(f)|}{\|f\|} = \sup_{\|f\|_{E'} = 1} |F(f)|.$$

For each $x \in E$, we define $F_x : E' \longrightarrow \mathbb{K}$ by:

$$F_x(f) = f(x) = \langle f, x\rangle.$$

We have the following result.

Theorem II.11. *The canonical mapping $\mathcal{F} : E \longrightarrow E''$ of the normed space E into its bidual, defined by $\mathcal{F}(x) = F_x$, is linear and isometric.*

Proof. First, let's show that $\mathcal{F}$ is linear.

- For all $x_1, x_2 \in E$,

$$\begin{aligned} \forall f \in E', \quad \langle \mathcal{F}(x_1 + x_2), f\rangle &= \langle F_{(x_1+x_2)}, f\rangle = f(x_1 + x_2) \\ &= f(x_1) + f(x_2) = F_{x_1}(f) + F_{x_2}(f) \\ &= \langle \mathcal{F}(x_1), f\rangle + \langle \mathcal{F}(x_2), f\rangle = \langle \mathcal{F}(x_1) + \mathcal{F}(x_2), f\rangle. \end{aligned}$$

Therefore, $\mathcal{F}(x_1 + x_2) = \mathcal{F}(x_1) + \mathcal{F}(x_2)$.

- For $f \in E'$, $x \in E$, and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}\langle \mathcal{F}(\lambda x), f\rangle &= \langle F_{\lambda x}, f\rangle = f(\lambda x) = \lambda f(x) \\ &= \lambda\langle F_x, f\rangle = \langle \mathcal{F}(x), f\rangle = \langle \lambda \mathcal{F}(x), f\rangle.\end{aligned}$$

Therefore, $\mathcal{F}(\lambda x) = \lambda \mathcal{F}(x)$.

Next, let's prove that F is an isometry.
For any $x \in E$,

$$\|F(x)\| = \sup_{f\neq 0,\, f\in E'} \frac{|F_x(f)|}{\|f\|} = \sup_{f\neq 0,\, f\in E'} \frac{|f(x)|}{\|f\|} \leq \sup_{f\neq 0,\, f\in E'} \frac{\|f\|\|x\|}{\|f\|} = \|x\|.$$

On the other hand, according to Corollary (II.2), for every $x \in E$, there exists $f_x \in E'$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$.

$$\|F(x)\| \geq |F(x)(f_x)| = \|x\|.$$

Hence, $\|F(x)\| = \|x\|$, and F is an isometry. $\square$

Definition II.10. *A space E is said to be reflexive if* $\operatorname{Im} F = E''$. *We write $E = E''$ and say that E and E'' are equivalent.*

Example II.12. $L^p(E)$, *where $1 < p < \infty$, is reflexive.*

$$(L^p(E))' = (L^q(E)), \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad (L^q(E))' = L^p(E).$$

Thus, $(L^p)''(E) = L^p(E)$.

Theorem II.12. *A Banach space is reflexive if and only if its bidual is reflexive.*

Proof. See [9]. $\square$

Reflexivity of Hilbert Spaces

Theorem II.13. *Every Hilbert space is reflexive.*

Proof. We will show that the application $J : E \to E''$ defined by

$$J(x) = f(x), \quad \forall f \in E'$$

is surjective.
Let H be a Hilbert space and $G \in H''$. For every vector $a \in H$, let L_a denote the linear form on H defined by $L_a(x) = \langle x, a \rangle$. Define $\varphi(a) = G(L_a)$, then φ is a continuous linear form on H. By the Riesz theorem, there exists $y \in H$ such that $\varphi = L_y$.
Let $f \in H'$ such that $f(a) = L_z(a)$ for all $a \in H$. Then, we have

$$G(f) = G(L_z) = \varphi(z) = \langle y, z \rangle = f(y) = J(y)(f).$$

Thus, J is surjective on H''. □

Reflexivity of Finite-Dimensional Spaces

Theorem II.14. *Every finite-dimensional normed vector space is reflexive.*

Proof. If E is a finite-dimensional space, we have:

$$\dim(E) = \dim(E') = \dim(E''),$$

On the other hand, the linear application J from E to E'', which is injective, is therefore bijective. As a result, E is reflexive. □

II.8 Conjugate operator

Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be two normed spaces over the same field $\mathbb{K}$, and let $\mathcal{L}(E, F)$ denote the normed space of linear and continuous operators.

Theorem II.15. *For any $A \in \mathcal{L}(E, F)$ and any continuous linear functional $y' \in F'$, the mapping*

$$\begin{aligned} x' \ : \quad & E \to \mathbb{K} \\ & x \mapsto x'(x) = y'(A(x)) \end{aligned}$$

is a continuous linear functional.

Proof. x' is linear since for all $\alpha, \beta \in \mathbb{K}$ and $x, y \in F$, we have

$$\begin{aligned} x'(\alpha x + \beta y) &= y'(A(\alpha x + \beta y)) \\ &= \alpha\, y'(A(x)) + \beta\, y'(A(y)) \\ &= \alpha\, x'(x) + \beta\, x'(y). \end{aligned}$$

Moreover,

$$|x'(x)| = |y'A(x)| \leq \|y'\|\|A\|\|x\|,$$

which gives

$$\|x'\| \leq \|y'\|\|A\|.$$

i.e., x' is bounded. □

Theorem II.16. *The map $A' : F' \longrightarrow E'$ given by*

$$A'y' = x'$$

is a continuous linear operator.

Proof. It is evident that A' is linear.
Moreover, for all $x \in E, y' \in F'$;

$$|A'(y')(x)| = |y'A(x)| \leq \|y'\|\|A\|\|x\|,$$

thus

$$\|A'(y')\| \leq \|y'\|\|A\|.$$

We conclude that A' is continuous and that $\|A'\| \leq \|A\|$. □

Definition II.11. *The operator A', as defined, is called the conjugate operator or adjoint of the operator A.*

Theorem II.17. *Let $A \in \mathcal{L}(E, F)$. Then*

$$\|A\| = \|A'\|.$$

Proof. We have already seen that $\|A'\| \leq \|A\|$.
Moreover, if $A = 0$, then $A' = 0$.
Assuming that $A \neq 0$, for all $x \in E$ and $y' \in F'$, we have

$$|A'(y')(x)| = |y'(A(x))| \leq \|A'\|\|y'\|\|x\|. \tag{II.8.1}$$

Let us fix $x \in E$ such that $A(x) \neq 0$.
According to the Hahn-Banach theorem, there exists $y' \in F'$ such that

$$\|y'\| = 1 \text{ and } y'(A(x)) = \|A(x)\|.$$

From relation (II.8.1), we obtain

$$\|A(x)\| \leq \|A'\|\|x\|.$$

Thus, $\|A\| \leq \|A'\|$. □

Properties

Let E, F, and G be normed spaces over the same field $\mathbb{K}$. Then the following properties hold.

1- For all $A, B \in \mathcal{L}(E, F)$,

$$(A + B)' = A' + B'.$$

2- For all $A \in \mathcal{L}(E, F)$ and $B \in \mathcal{L}(F, G)$,

$$(A \circ B)' = B' \circ A'.$$

3- For all $\lambda \in \mathbb{K}$ and $A \in \mathcal{L}(E, F)$,

$$(\lambda A)' = \lambda A'.$$

4- If I is the identity operator from $E \longrightarrow E$, then $I' : E' \longrightarrow E'$ satisfies

$$I'(y') = y', \quad \forall y' \in E'.$$

5- If $A \in \mathcal{L}(E, F)$ and $A^{-1} \in \mathcal{L}(F, E)$, then $(A^{-1})'$ exists and

$$(A^{-1})' = (A')^{-1}.$$

Proof. 1- For all $x \in E$ and $y' \in F'$, we have

$$\begin{aligned} (A+B)'(y')(x) &= (y')(A+B)(x) = y'(A(x)) + y'(B(x)) \\ &= A'(y')(x) + B'(y')(x) \\ &= \big(A'(y') + B'(y')\big)(x). \end{aligned}$$

Thus, $(A+B)'(y') = A'(y') + B'(y')$ for all $y' \in F'$. Since y' is arbitrary, it follows that

$$(A + B)' = A' + B'.$$

2-

$$\begin{aligned}(A \circ B)'(y')(x) &= (y')(A \circ B)(x) = y'\big(A(B(x))\big) \\ &= A'(y')(B(x)) = (A' \circ B')(y')(x),\end{aligned}$$

for all $x \in E$ and $y' \in F'$. Thus, $(A \circ B)' = B' \circ A'$.

3-

$$(\lambda A)'(y')(x) = y'(\lambda A(x)) = \lambda y'(A(x)) = \lambda A'(y')(x),$$

for all $\lambda \in \mathbb{K}$, $y' \in F'$, and $x \in E$.

4-

$$I'(y'(x)) = y'(I(x)) = y'(x),$$

thus I' is the identity operator on E'.

5- Assume that $A^{-1} \in \mathcal{L}(F, E)$. We will show that A' is injective. Let $A'(y'_1)(x) = A'(y'_2)(x)$. For all $x \in E$, we have

$$0 = A'(y'_1 - y'_2)(x) = (y'_1 - y'_2)(A(x)).$$

This implies $\langle y'_1 - y'_2, y\rangle = 0, \quad \forall y \in F$. Thus, $y'_1 = y'_2$ and A' is injective.

6- We will show that A' is surjective. Since $A^{-1} \in \mathcal{L}(E, F)$, there exists $(A^{-1})'$. Now, for every $x \in E$ such that $x = A^{-1}y$ and $y \in F$, and for $f \in E'$, we have

$$\begin{aligned}f(x) &= f(A^{-1})(y) = (A^{-1})'f(y) \\ &= \big((A^{-1})' \circ f\big)(Ax) = \big((A^{-1})' \circ f(Ax)\big) \\ &= A'\big((A^{-1})'f\big)(x).\end{aligned}$$

Thus,

$$f = A'\big((A^{-1})'f\big), \tag{II.8.2}$$

i.e., $f \in R(A') = E'$ and A' is surjective.
From relation (II.8.2), we have

$$A'\big((A^{-1})' = I_{E'} \quad \text{and} \quad (A^{-1})' = (A')^{-1}.$$

□

Theorem II.18. *Let $\{A_n;\ n \in \mathbb{N}\} \subset \mathcal{L}(E, F)$ be a sequence that converges in norm to $A \in \mathcal{L}(E, F)$. Then the sequence $\{A'_n;\ n \in \mathbb{N}\}$ converges in norm to A'.*

Proof. We have,

$$\|A'_n - A'\| = \|(A_n - A)'\| = \|A_n - A\|.$$

$\square$

II.9 Compact operators

Definition II.12. *A linear operator $A : E \longrightarrow F$ is compact if it transforms every bounded set M into a relatively compact set in F.*
i.e. $\forall M \subset E$, M bounded; $\overline{A(M)}$ is compact.
We denote by $K(E, F)$ the set of compact operators.

Remark II.3. *Let $A : E \longrightarrow F$ be compact. Then A is continuous.*
Indeed, let M be a bounded set in E. Then, $A(M) \subset \overline{A(M)}$, so $A(M)$ is precompact, and consequently, $A(M)$ is bounded. That is, if A is compact, then A is continuous.

Lemma II.3. *Let E, F be two normed spaces and $A : E \longrightarrow F$ a linear operator.*
A is compact if and only if from every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in E, one can extract a convergent subsequence from the sequence $(Au_n)_{n\in\mathbb{N}}$ in F.

Proof. Let M be a bounded set in E and $(v_n)_{n\in\mathbb{N}} \subset \overline{A(M)}$. Then, there exists $(w_n)_{n\in\mathbb{N}} \subset A(M)$ such that

$$\|v_n - w_n\| < \frac{1}{n}, \quad w_n = A(u_n) \text{ and } (u_n)_{n\in\mathbb{N}} \subset M.$$

Thus, $(w_n)_{n\in\mathbb{N}}$ has a convergent subsequence $\{Au_{n_k}\}$ such that

$$w_{n_k} = Au_{n_k} \longrightarrow v \in F.$$

Therefore,

$$\|v_{n_k} - v\| \leq \|v_{n_k} - w_{n_k}\| + \|w_{n_k} - v\| \underset{k\longrightarrow+\infty}{\longrightarrow} 0,$$

which implies $v_{n_k} \underset{k}{\longrightarrow} v$ and $\overline{A(M)}$ is compact.
Conversely, let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence. We have $\{Au_n\}_n \subset \overline{\{Au_n\}_n}$, which is relatively compact, so there exists a subsequence such that $Au_{n_k} \to v \in F$. $\square$

Proposition II.7. *A is compact if and only if $A(B)$ is relatively compact.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B. Then, it is bounded and, by the lemma (II.3), there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $Au_{n_k} \underset{k}{\longrightarrow} v \in \overline{A(B)}$, thus $A(B)$ is relatively compact.
Conversely, Let $A(B)$ be relatively compact. If $\{u_n\}$ is bounded, then $\|u_n\| \leq M$ for all $n \in \mathbb{N}$, and $\left\|\frac{u_n}{M}\right\| \leq 1$.
Thus, $\left\{\frac{u_n}{M}\right\}_n \subset B$ and $A\left(\frac{u_n}{M}\right)_n \subset A(B)$, which gives

$$A\left(\frac{u_{n_k}}{M}\right) \longrightarrow v \in F \text{ and } A(u_{n_k}) \underset{k}{\longrightarrow} Mv \text{ in } F.$$

Hence A is compact. □

Definition II.13. *Let $C(E,F)$ be the set of all continuous functions from the metric space (E,d) to the metric space (F,ρ) and $\mathfrak{F} \subset C(E,F)$. We say that $\mathfrak{F}$ is equicontinuous at x if*

$$\forall \epsilon, \exists \delta(x,\epsilon) > 0, \forall y \in E,\ d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \epsilon,\ \forall f \in \mathfrak{F}.$$

Theorem II.19. *[9] [Arzelà-Ascoli Theorem] Let (E,d) be a metric space and (F,ρ) a complete metric space. Then, a subspace $\mathfrak{F} \subset C(E,F)$ is relatively compact if and only if $\mathfrak{F}$ is equicontinuous and, for every $x \in E$,*

$$\mathcal{F}(x) = \{f(x) \subset F,\ \textit{for all}\ f \in \mathfrak{F}\}$$

is relatively compact.

Corollary II.3. *[9] Let E be a compact space. A space $\mathfrak{F} \subset C(E,\mathbb{R}^N)$ is relatively compact if and only if $\mathfrak{F}$ is equicontinuous and uniformly bounded.*

Example II.13. *Let $f_n : [a,b] \longrightarrow \mathbb{R}$, $n = 1, \dots,$ be continuous functions that satisfy $|f_n| \leq M$ and $\|f_n\| \leq k$ on $[a,b]$. Then,*

$$\forall n,\ \forall x,y \in [a,b];\ |f_n(x) - f_n(y)| \leq k|x-y|$$

and

$$\forall \epsilon > 0,\ \exists \delta = \frac{\epsilon}{k},\ \forall x,y \in [a,b];\ |x-y| < \frac{\epsilon}{k} \Rightarrow |f_n(x) - f_n(y)| < \epsilon,\ n = 1, \dots$$

i.e. $\{f_n\}_n$ is equicontinuous and uniformly bounded, hence relatively compact.

Example II.14. *Let* $U = C\big([0,1],\mathbb{R}\big)$, $K : U \longrightarrow U$,

$$Ku(x) = \int_0^1 k(x,y)u(y)\,dy,$$

where $k(x,y)$ *is a continuous function on* $0 \leq x, y \leq 1$. *We want to show that* K *is compact. Let* $S_N \subset U$ *be a bounded set. Then,* $\forall u \in S_N$; $\|u\| = \sup_{x\in[0,1]} |u(x)| \leq N$ *and for every* $x \in [0,1]$, $u \in S_N$, *we have*

$$\begin{aligned} |Ku(x)| &= \left| \int_0^1 k(x,y)u(y)\,dy \right| \\ &\leq \int_0^1 |k(x,y)||u(y)|\,dy \leq MN. \end{aligned}$$

So $\|Ku\| \leq MN$ *and* $K(S_N)$ *is uniformly bounded.*
On the other hand,

$$\forall x_1, x_2 \in [0,1],\ u \in S_N,\ |x_1 - x_2| < \delta,$$

$$\begin{aligned} |Ku(x_1) - Ku(x_2)| &\leq \int_0^1 |k(x_1,y) - k(x_2,y)||u(y)|\,dy \\ &\leq \frac{\epsilon}{N}\int_0^1 |u(y)|\,dy \leq \frac{\epsilon}{N}N. \end{aligned}$$

Thus, $\{Ku, u \in S_N\}$ *is equicontinuous, which implies that* $K(S_N)$ *is relatively compact and consequently,* K *is compact.*

Theorem II.20. 1 *Let* E *and* F *be two normed spaces and* $A_1, A_2 : E \longrightarrow F$ *be compact operators. Then,* $A = \alpha A_1 + \beta A_2$, *for all* $\alpha, \beta \in \mathbb{K}$, *is also compact.*

2 *Let* E, F, *and* G *be normed spaces and* $A : E \longrightarrow F$ *and* $B : F \longrightarrow G$ *be linear and continuous operators. If one of the operators is compact, then the composition* $B \circ A : E \longrightarrow G$ *is also compact.*

3 *Let* F *be a Banach space. If the sequence of compact operators* $\{A_n : E \longrightarrow F\}$, $n = 1, 2, \ldots$, *converges in norm to the operator* $A : E \longrightarrow F$, *then* A *is compact.*

Proof. 1 Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Then, there exists a subsequence $\{u_{\varphi(k)}\}$ such that $A_1(u_{\varphi(k)}) \longrightarrow v_1$ in F and $A_2(u_{\psi(k)}) \longrightarrow v_2$ in F, hence $A(u_{\psi(\varphi(k))}) = \alpha A_1(u_{\psi(\varphi(k))}) + \beta A_2(u_{\psi(\varphi(k))}) \longrightarrow \alpha v_1 + \beta v_2$ Therefore, A is compact.

2 Let A be compact and B continuous. If S is a bounded set in E, then $\overline{A(S)}$ is compact and, since B is continuous, $B(\overline{A(S)})$ is compact and also closed. The continuity of B implies

$$B(\overline{A(S)}) \subset \overline{B(A(S))}. \tag{II.9.1}$$

On the other hand,

$$A(S) \subset \overline{A(S)} \Rightarrow B(A(S)) \subset B(\overline{A(S)}).$$

Thus,

$$\overline{B(A(S))} \subset B(\overline{A(S)}). \tag{II.9.2}$$

From equations (II.9.1) and (II.9.2), we obtain $\overline{B(A(S))} = B(\overline{A(S)})$ and $B \circ A$ is compact. Now, we assume that B is compact and A is linear and continuous. If S is bounded, then $A(S)$ is bounded. Since B is compact, $\overline{B(A(S))}$ is compact, so $B \circ A$ is compact.

3 Let $A(x) = \lim_n A_n(x)$, for all $x \in E$. Then A is linear and continuous.

$$\text{If } A_n \longrightarrow A, \text{ then } \forall \epsilon > 0,\ \exists N(\epsilon) > 0;\ \|A_n - A\| < \frac{\epsilon}{N},\ n \geq N.$$

Let B be the closed unit ball. We want to show that $A(B)$ is relatively compact. We have, $\|A_n(x) - A(x)\| \leq \|A_n - A\|\|x\| \leq \|A_n - A\| < \epsilon$, $\forall x \in B$, $n \geq N$. Since A_N is compact, $A_N(B)$ is compact, hence pre-compact. This gives $A_N(B) \subset \bigcup_{i=1}^n B(A_N(x_i), \frac{\epsilon}{3}), \{x_i, i = 1, \ldots, n\} \subset B$. Let $x \in B$. Then, $A_n(x) \in B(A_N(x_k), \frac{\epsilon}{3})$ and we have

$$\begin{aligned} \|A(x) - A(x_k)\| &\leq \|A(x) - A_N(x)\| + \|A_N(x) - A_N(x_k)\| \\ &+ \|A_N(x_k) - A(x_k)\| < \epsilon. \end{aligned}$$

Hence $A(B)$ is precompact and $\overline{A(B)} \subset \bigcup_{i=1}^n \overline{B_f(x_k, \epsilon)}$. Therefore, $\overline{A(B)}$ is precompact and closed in a complete space, so it is compact.

□

II.10 Finite-rank operator

Definition II.14. *A linear operator $T : E \to F$ between two vector spaces E and F is called a **finite-rank operator** if its image $T(E)$ is a finite-dimensional subspace of F.*
In other words, T is a finite-rank operator if there exists a finite set of vectors $\{y_1, y_2, \ldots, y_n\} \subset F$ such that any element $T(x)$ for $x \in E$ can be expressed as a linear combination of $y_1, y_2, \ldots, y_n$.

Example II.15. *Consider the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by*

$$T(x, y, z) = (x + y, y + z).$$

To see that T is a finite-rank operator, observe that the image of T is spanned by the vectors

$$T(1, 0, 0) = (1, 0) \quad \textit{and} \quad T(0, 1, 0) = (1, 1).$$

Since the span of these two vectors is a finite-dimensional subspace of $\mathbb{R}^2$, with dimension at most 2, the operator T is of finite rank.

Proposition II.8. *[7] Every bounded finite-rank operator is compact.*

Proof. Let $T \in \mathcal{L}(E, F)$ be a finite-rank operator. Since T is continuous, for every $x \in B_E$, we have $\|T(x)\| \leq \|T\|$. Therefore, $T(B_E)$ is bounded in F, and consequently, $\overline{T(B_E)}$ is also bounded. Moreover, the image $\mathrm{Im}(T)$ is closed because it is a finite-dimensional vector space. Hence, we have

$$\overline{T(B_E)} \subset \overline{\mathrm{Im}(T)} = \mathrm{Im}(T).$$

Finally, $\overline{T(B_E)}$ is closed and bounded in a finite-dimensional space, and therefore, it is compact. □

Theorem II.21. *[7] Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$, and let $T \in K(E, H)$. Then, there exists a sequence $\{T_n\}_{n \in \mathbb{N}}$ of finite-rank operators in $\mathcal{L}(E, H)$ that converges to T in $\mathcal{L}(E, H)$.*

Proof. Let $n \in \mathbb{N}$ be fixed. If T is compact, then $T(B_E)$ is relatively compact, and there exists $k_n \in \mathbb{N}^*$ and $y_1, \ldots, y_{k_n} \in H$, such that

$$\overline{T(B_E)} \subset \bigcup_{i=1}^{k_n} B(y_i, \frac{1}{n+1}).$$

Let $L_n = \{y_1, \ldots, y_{k_n}\}$ be a finite-dimensional vector space, which is closed in the Hilbert space H, and thus $H = L_n \oplus L_n^{\perp}$.
Now, let $P_n \in \mathcal{L}(H)$ be the orthogonal projection of H onto L_n. Then, $T_n = P_n T \in \mathcal{L}(E, H)$ is a finite-rank operator, because

$$\dim(\mathrm{Im}\, T_n) = \dim(\mathrm{Im}(P_n \circ T)) \leq \dim(L_n) \leq k_n.$$

For all $x \in B_E$, there exists $i_0 \in \{1, \ldots, k_n\}$, such that

$$\|Tx - y_{i_0}\| < \frac{1}{n+1}.$$

But $y_{i_0} \in L_n$, and $P_n Tx$ is the orthogonal projection of Tx onto L_n, so

$$\|Tx - P_n Tx\| = \operatorname{dist}(Tx, L_n) \leq \|Tx - y_{i_0}\| < \frac{1}{n+1}.$$

Thus, we obtain

$$\forall x \in B_E, \; \|Tx - T_n x\| < \frac{1}{n+1},$$

which implies

$$\|T - T_n\| < \frac{1}{n+1}, \quad \text{for all } n \in \mathbb{N}^*.$$

□

II.11 Spectral Properties of Compact Operators

Reminder: Let M be a closed subspace of a normed vector space E. We say that M has a topological supplement if there exists a closed subspace N of E such that $E = M \oplus N$.

Remark II.4. *If M has a topological supplement N, then both M and N are closed in E. Moreover, if M is finite-dimensional, then M has a topological supplement.*

Lemma II.4. *If $T \in K(E)$, then $M = \ker(I - T)$ has finite dimension. In particular, according to remark (II.4), M has a topological supplement N.*

Proof. First, let us note that $B_F = T(B_F)$. Indeed, for every $y \in M$, we have $y = T(y)$. Since $M = (\ker(I - T))^{-1}\{0\}$ is closed in E, and $B_F = B_E \cap F$, it is also closed in E. Moreover, since $B_F \subset B_E$, we have

$$B_F = T(B_F) \subset \overline{T(B_F)}.$$

So B_F is closed and contained in $\overline{T(B_F)}$, which is a compact set in E because $T \in K(E)$. Therefore, B_F is a compact set in F, and by Riesz Theorem, we conclude that A is finite-dimensional. □

Lemma II.5. *Let $T \in K(E)$, $M = \ker(I-T)^{-1}\{0\}$, and let N be a topological complement of M. Then, the operator $S = (I-T)|_N$ is linear, continuous, and injective. Moreover, there exists a constant $C > 0$ such that, for all $x \in N$, we have*

$$\|Sx\| \geq C\|x\|. \tag{II.11.1}$$

Remark II.5. *According to Corollary (II.2), we deduce that $\overline{Im(S)} = Im(S)$.*

Proof. The map $I - T$ is linear and continuous, and thus S is also linear and continuous. Moreover, if $x \in \ker(S)$, then $x \in N$ and $(I-T)(x) = 0$, which implies $x \in M \cap N = \{0\}$, so S is injective.
Suppose the inequality (II.11.1) is false. Then, for every $n \in \mathbb{N}^*$, there exists $x_n \in N$, such that

$$\|Sx_n\| < \frac{1}{n}\|x_n\|. \tag{II.11.2}$$

It is evident that $\|x_n\| \neq 0$. For $n \in \mathbb{N}^*$, define

$$y_n = \frac{x_n}{\|x_n\|} \in B_E \cap N.$$

Since $T \in K(E)$, there exists a subsequence Ty_{n_k} and $z \in E$ such that

$$Ty_{n_k} \underset{k\to\infty}{\longrightarrow} z.$$

Now, from (II.11.2), we have

$$\|Sy_{n_k}\| < \frac{1}{n_k} \text{ for all } n_k \in \mathbb{N}^*.$$

So, that

$$y_{n_k} = Sy_{n_k} + Ty_{n_k} \underset{k\to\infty}{\longrightarrow} z. \tag{II.11.3}$$

Moreover, since for all $k \in \mathbb{N}^*$, $y_{n_k} \in N$, we deduce $z \in \overline{N} = N$.
The continuity of S on N implies that

$$Sy_{n_k} \underset{k\to\infty}{\longrightarrow} Sz.$$

From (II.11.3), we obtain $Sz = 0$. But since S is injective, we conclude $z = 0$, which is a contradiction because

$$\|z\|_E = \lim_{k\to\infty} \|y_{n_k}\| = 1.$$

Hence, the desired result. □

Proposition II.9. *If $T \in K(E)$, then $\operatorname{Im}(I-T)$ is closed in E.*

Proof. Let $M = \ker(I-T)$ and N be the topological complement of M. Suppose that $z \in \overline{\operatorname{Im}(I-T)}$. Then, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in E such that $z_n - Tz_n \underset{n\to\infty}{\longrightarrow} z$. Since $E = M \oplus N$, for each $n \in \mathbb{N}$, there exist $(x_n, y_n) \in M \times N$ such that $z_n = x_n + y_n$. Let $S = \ker(I-T)|_N$. Since for each $n \in \mathbb{N}$, $x_n \in M = \ker(I-T)$, i.e., $x_n - Tx_n = 0$, we have $Sy_n = y_n - Ty_n = z_n - Tz_n - (x_n - Tx_n) \underset{n\to\infty}{\longrightarrow} z$. By Remark (II.5), we deduce

$$z \in \overline{\operatorname{Im}(S)} = \operatorname{Im}(S) \subset \operatorname{Im}(I-T). \tag{II.11.4}$$

Hence, the desired result. □

Theorem II.22. *Let $T \in K(E)$. Then, $(I-T)$ is injective if and only if $(I-T)$ is invertible.*

Theorem II.23. *Let $T \in K(E)$.*

1. *If E is finite-dimensional, then $0 \in \sigma(T)$.*
2. *$V_p(T) - \{0\} = \sigma(T) - \{0\}$, and for $\lambda \in \sigma(T) - \{0\}$, the associated eigenspace $\ker(I-T)$ is finite-dimensional.*

Proof.

1. If $0 \notin \sigma(T)$, then T is invertible in $\mathcal{L}(E)$ and $I = T^{-1}T \in K(E)$. Therefore, $I(B_E) = B_E$ is compact, and by the Riesz theorem, E is finite-dimensional.

2. $\lambda \in \sigma(T) - \{0\}$ if and only if $\lambda \neq 0$ and $(\lambda I - T)$ is not invertible. This is equivalent to $\lambda \neq 0$ and $(I - \lambda^{-1}T)$ is not invertible in $\mathcal{L}(E)$. Now, by Theorem (II.22) applied to $\lambda^{-1}T$, $(I - \lambda^{-1}T)$ is not invertible if and only if $(I - \lambda^{-1}T)$ is not injective. Consequently, $\lambda \in \sigma(T)\backslash\{0\}$ if and only if $\lambda \neq 0$ and $\lambda \in V_p(T)\backslash\{0\}$.

   Furthermore, by Lemma (II.4), $\ker(I-\lambda^{-1}T)$ is finite-dimensional. Now, $\dim\ker(I-\lambda^{-1}T) = \dim\ker(\lambda I-T)$, thus $\ker(\lambda I-T)$ is finite-dimensional.

□

Corollary II.4 (Fredholm Alternative). *Let $T \in K(E)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. We consider the Fredholm equation: for $y \in E$, find $x \in E$ such that*

$$\lambda x - Tx = y. \tag{II.11.5}$$

Then, two cases are possible:

1. *For every y, there exists a unique $x \in E$ solution of (II.11.5).*
2. *The homogeneous equation $\lambda x - Tx = 0$ has a nontrivial solution $x \neq 0$.*

Proof. If $\lambda \notin V_p(T)$, then $\lambda I - T$ is injective and therefore invertible, which implies the uniqueness of the solution to (II.11.5). If $\lambda \in V_p(T)$, there exists $x_0 \in E \setminus \{0\}$ such that $\lambda x_0 - Tx_0 = 0$. $\square$

II.12 Adjoint Operator in a Hilbert Space

We denote by $(H, \langle\cdot,\cdot\rangle)$, $(H_1, \langle\cdot,\cdot\rangle)$, and $(H_2, \langle\cdot,\cdot\rangle)$ Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorem II.24. *Let $T \in \mathcal{L}(H_1, H_2)$. Then, for every $y \in H_2$, there exists $z \in H_1$ such that*

$$\forall x \in H_1, \quad \langle Tx, y\rangle = \langle x, z\rangle.$$

We denote $z = T^*y$.

Definition II.15. *The operator T^*, defined from H_2 into H_1 by*

$$\forall x \in H_1,\ \forall y \in H_2;\ \langle Tx, y\rangle = \langle x, T^*y\rangle,$$

is called the adjoint of T.

Proof. Fix $y \in H_2$. The mapping

$$\varphi_y : H_1 \to \mathbb{K}, \quad x \mapsto \varphi_y(x) = \langle Tx, y\rangle$$

is linear, and we have

$$\forall x \in H_1;\ |\varphi_y(x)| = |\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|.$$

Thus, φ_y is continuous. By the Riesz representation Theorem, there exists $z \in H_1$ such that

$$\langle \varphi_y, x\rangle = \langle x, z\rangle.$$

Therefore,

$$\langle Tx, y\rangle = \langle x, z\rangle = \langle x, T^*y\rangle.$$

□

Theorem II.25. *Let $T \in \mathcal{L}(H_1, H_2)$. Then $T^* \in \mathcal{L}(H_2, H_1)$.*

Proof.

- Let y_1, y_2 and $\alpha, \beta \in \mathbb{K}$. Then, for all $x \in H_1$,

$$\begin{aligned}\langle x, T^*(y_1 + y_2)\rangle &= \langle Tx, y_1 + y_2\rangle \\ &= \langle Tx, y_1\rangle + \langle Tx, y_2\rangle \\ &= \langle x, T^*y_1\rangle + \langle x, T^*y_2\rangle.\end{aligned}$$

  Thus, $T^*(y_1 + y_2) = T^*y_1 + T^*y_2$.

- For all $x \in H_1$, $\lambda \in \mathbb{K}$,

$$\begin{aligned}\langle x, T^*(\lambda y)\rangle &= \langle Tx, \lambda y\rangle \\ &= \bar{\lambda}\langle Tx, y\rangle \\ &= \bar{\lambda}\langle x, T^*y\rangle = \langle x, \lambda T^*y\rangle.\end{aligned}$$

  Furthermore, for all $y \in H_2$, we have

$$\begin{aligned}\|T^*y\|^2 &= \langle T^*y, T^*y\rangle = \langle T(T^*(y)), y\rangle \\ &\leq \|T(T^*(y))\|\|y\| \leq \|T\|\|T^*(y)\|\|y\|.\end{aligned}$$

  Thus, $\|T^*y\| \leq \|T\|\|y\|$, which gives

$$\|T^*\| \leq \|T\|.$$

  On the other hand,

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle = \langle x, T^*(T(y))\rangle \\ &\leq \|T(T^*(x))\|\|x\| \leq \|x\|\|T^*\|\|Tx\|.\end{aligned}$$

  Hence, $\|Tx\| \leq \|T^*\|\|x\|$, and therefore,

$$\|T\| \leq \|T^*\|.$$

  From the above two inequalities, we conclude that

$$\|T\| = \|T^*\|.$$

□

Proposition II.10. *Let $S, T \in \mathcal{L}(H_1, H_2)$, $u \in \mathcal{L}(H_2, H)$, and $\lambda, \beta \in \mathbb{K}$, we have*

(1) $I^*_{H_1} = I_{H_1}$ $\big(I_{H_1} \in \mathcal{L}(H_1)\big)$.

(2) $(\lambda S + \mu T)^* = \bar{\lambda} S^* + \bar{\mu} T^*$.

(3) $U \circ T \in \mathcal{L}(H_1, H)$, $(U \circ T)^* = T^* \circ U^*$.

(4) $(T^*)^* = T$.

(5) $\|T^* T\| = \|T^* T\| = \|T\|$.

Proof. (1) For $x, y \in H_1$, $\langle Ix, y \rangle = \langle x, I^*(y) \rangle = \langle x, Iy \rangle$. Thus, $I^*_{H_1} = I$.

(2) For all $x, y \in H_2$,

$$\begin{aligned} \langle x, (\lambda S + \mu T)^*(y) \rangle &= \langle (\lambda S + \mu T)(x), y \rangle \\ &= \lambda \langle S(x), y \rangle + \mu \langle T(x), y \rangle \\ &= \big\langle x, \bar{\lambda} S^*(y) + \bar{\mu} T^*(y) \big\rangle . \end{aligned}$$

Thus, $(\lambda S + \mu T)^* = \bar{\lambda} S^* + \bar{\mu} T^*$.

(3) For $x, y \in H_2$,

$$\langle y, T(x) \rangle = \langle T^*(y), x \rangle = \langle y, (T^*)^*(x) \rangle .$$

Thus, $T = (T^*)^*$.

(4) For $x \in H_1$ and $y \in H$, we have

$$\begin{aligned} \langle x, (U \circ T)^*(y) \rangle &= \langle U \circ T(x), y \rangle \\ &= \langle T(x), U^*(y) \rangle \\ &= \langle x, T^* \circ U^*(y) \rangle . \end{aligned}$$

This implies, $(U \circ T)^* = T^* \circ U^*$.

(5) On one hand,

$$\|T^* T\| \leq \|T^*\| \|T\| = \|T\|^2.$$

On the other hand,

$$\begin{aligned} \|Tx\|^2 &= \langle T(x), T(x) \rangle = \langle x, T^* T(x) \rangle \\ &\leq \|x\| \|T^* T\| \|x\|, \end{aligned}$$

which implies

$$\|T\|^2 \leq \|T^*T\|.$$

In conclusion,

$$\|T\|^2 = \|T^*T\|.$$

Furthermore,

$$\|TT^*\| = \|(T^*)^*T^*\| = \|T^*\|^2 = \|T\|^2.$$

□

Theorem II.26. *[7] An operator $T \in \mathcal{L}(H_1, H_2)$ is invertible if and only if T^* is invertible. Moreover, $(T^*)^{-1} = (T^{-1})^*$.*

Proof. Let $T \in \mathcal{L}(H_1, H_2)$ be invertible. Then,

$$(T^{-1})^* \circ T^* = (T \circ T^{-1})^* = I_{H_2}^* = I_{H_2}$$

and

$$T^* \circ (T^{-1})^* = (T^{-1} \circ T)^* = I_{H_1}^* = I_{H_1}.$$

Conversely, if $T^* \in \mathcal{L}(H_2, H_1)$, then $T = (T^*)^* \in \mathcal{L}(H_1, H_2)$ and T is invertible. □

II.13 Spectrum of an operator

Proposition II.11. *[7] Let E be a Banach space, $T \in \mathcal{L}(E)$, and $S = \sum_{n\geq 0} T^n$.*
If $\|T\| < 1$, then the operator $I - T \in GL(E)$, and we have

$$(I - T)^{-1} = \sum_{n\geq 0} T^n.$$

Moreover, $GL(E)$ is an open subset of $\mathcal{L}(E)$ and the map

$$T \mapsto (T)^{-1}$$

is continuous on $GL(E)$.

Proof. Since $\|T\| < 1$, the series $S = \sum_{n\geq 0} T^n$ is normally convergent, and therefore convergent because $\mathcal{L}(E)$ is complete.
Moreover,

$$(I-T)S = (IS - TS) = \sum_{n\geq 0} T^n - \sum_{n\geq 0} T^{n+1} = T^0 = I.$$

Similarly, $S(I-T) = I$.
We deduce that $S = (I-T)^{-1}$.

- Let's show that $GL(E)$ is an open set in $\mathcal{L}(E)$.
  Let $T_0 \in GL(E)$. Then, if $T \in GL(E)$ we obtain,

  $$T = T_0 + T - T_0 = T_0\big(I - (I - T_0^{-1}T)\big)$$

  and
  $$\|I - T_0^{-1}T\| \leq \|T_0^{-1}\|\|T_0 - T\|.$$
  Therefore, if $\|T_0 - T\| < \|T_0^{-1}\|^{-1}$, then $\big(I - (I - T_0^{-1}T)\big)$ is invertible, and T is invertible.
  In other words,
  $$B(T_0, \|T_0^{-1}\|^{-1}) \subset GL(E).$$
  So $GL(E)$ is an open set in $\mathcal{L}(E)$.
  Moreover, if $T \in B(T_0, \|T_0^{-1}\|^{-1})$, then

  $$T^{-1} = \big(I - (I - T_0^{-1}T)\big)^{-1} T_0^{-1} = \sum_{n\geq 0} (I - T_0^{-1}T)^n T_0^{-1}.$$

  Thus, we obtain

  $$\begin{aligned} \|T^{-1} - T_0^{-1}\| &= \|\sum_{n\geq 0}(I - T_0^{-1}T)^n T_0^{-1} - T_0^{-1}\| \\ &= \|\sum_{n\geq 1}(I - T_0^{-1}T)^n T_0^{-1}\| \\ &\leq \|T_0^{-1}\| \sum_{n\geq 1} \|(I - T_0^{-1}T)\|^n \\ &\leq \|I - T_0^{-1}T\|\|T_0^{-1}\| \frac{1}{1 - \|I - T_0^{-1}T\|} \\ &= \frac{\|T_0 - T\|\|T_0^{-1}\|^2}{1 - \|I - T_0^{-1}T\|} \overset{T\to T_0}{\longrightarrow} 0. \end{aligned}$$

  This shows the continuity of the mapping $T \mapsto T^{-1}$ on $GL(E)$.

$\square$

Definitions and Properties

Definition II.16 (Resolvent set)**.** *The **resolvent set** of T is defined as*

$$\rho(T) = \{\lambda \in \mathbb{K};\ (\lambda I - T) \text{ is invertible}\}.$$

*The elements of $\rho(T)$ are called the **resolvent values** of T.*

Definition II.17 (Resolvent operator)**.** *If $\lambda \in \rho(T)$, the **resolvent** is defined as*

$$R_\lambda(T) = (\lambda I - T)^{-1}.$$

The resolvent is often simply denoted by R_λ.

Definition II.18 (Spectrum)**.** *The **spectrum** $\sigma(T)$ is the set*

$$\sigma(T) = \mathbb{K} - \rho(T).$$

*The elements of $\sigma(T)$ are called the **spectral values** of T.*

Definition II.19 (Eigenvalues)**.** *We say that $\lambda \in \mathbb{K}$ is an **eigenvalue** of T if*

$$(\lambda I - T) \text{ is not injective.}$$

That is, the equation $Tx = \lambda x$ has nontrivial solutions $x \in E - \{0\}$. In other words, the set of eigenvalues $V_p(T)$ is given by:

$$V_p(T) = \{\lambda \in \mathbb{K};\ \ker(\lambda I - T) \neq \{0\}\}.$$

Remark II.6.

1. *The previous definitions remain valid even if E is not a Banach space.*
2. $V_p(T) \subset \sigma(T)$.
3. *The set of eigenvalues is also called the point spectrum.*
4. *If E is finite-dimensional,*

$$(\lambda I - T) \text{ is invertible if and only if } \ker(\lambda I - T) = \{0\}.$$

Proposition II.12. *Let $T \in \mathcal{L}(E)$.*

1. *If $|\lambda| \geq \|T\|$, then $\lambda \in \rho(T)$.*
2. *$\rho(T)$ is a non-empty open subset of $\mathbb{K}$.*
3. *$\sigma(T)$ is compact.*
4. *$\overline{V_p(T)} \subset \sigma(T)$.*

Proof.

1. If $|\lambda| \geq \|T\|$, then $\left\|\frac{T}{\lambda}\right\| < 1$ and $(I - \lambda^{-1}T)$ is invertible. Thus,
$$\left[\lambda^{-1}(\lambda I - T)\right]^{-1} = \lambda\left[\lambda I - T\right]^{-1} = \left[I - \lambda^{-1}T\right],$$
i.e., $\lambda \in \rho(T)$.

2. From 1., $\rho(T)$ is non-empty.
Let φ be a map from $\mathbb{K}$ to $\mathcal{L}(E)$, defined by $\varphi : \lambda \longmapsto (\lambda I - T)$. Then we have
$$\varphi^{-1}(GL(E)) = \rho(T).$$
Furthermore, for all $\lambda, \mu \in \mathbb{K}$,
$$\|\varphi(\lambda) - \varphi(\mu)\| = \|(\lambda - \mu)T\| \leq |\lambda - \mu|\|T\|.$$
Thus, φ is continuous.

3. $\sigma(T) = \mathbb{K} - \rho(T)$, so $\sigma(T)$ is closed, and from (1), it is also bounded in $\mathbb{K}$ (since $\mathbb{K}$ is finite-dimensional). Therefore, $\sigma(T)$ is compact.

4. From Remark (II.6), $V_p(T) \subset \overline{\sigma(T)} = \sigma(T)$, which gives $\overline{V_p(T)} \subset \sigma(T)$.

$\square$

Proposition II.13 (Resolvent identity)**.** *Let $T \in \mathcal{L}(E)$ and $\lambda, \mu \in \rho(T)$. Then,*
$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu = (\mu - \lambda)R_\mu R_\lambda.$$
Moreover, the map $\lambda \longmapsto R_\lambda$ is differentiable on $\rho(T)$, and its derivative is given by
$$\frac{dR_\lambda}{d\lambda} = -R_\lambda^2.$$

Proof. Let $R_\lambda = (\lambda I - T)^{-1}$.

$$\begin{aligned} R_\lambda &= R_\lambda(\mu I - T)R_\mu = R_\lambda[\lambda I - T + (\mu - \lambda)I]R_\mu \\ &= [(I + (\mu - \lambda)R_\lambda]\, R_\mu = R_\mu + (\mu - \lambda)\lambda)R_\lambda R_\mu. \end{aligned}$$

Thus,

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu.$$

The map $\lambda \longmapsto R_\lambda$ from $\rho(T)$ to $GL(E)$ is continuous (it is the composition of two continuous functions, φ and $T \longmapsto T^{-1}$).
Furthermore, for all $h > 0$, we have

$$\frac{1}{h}[R_{\lambda+h} - R_\lambda] = -R_{\lambda+h}R_\lambda.$$

Thus, the continuity of $\lambda \longmapsto R_\lambda$ implies the differentiability and

$$\frac{dR_\lambda}{d\lambda} = -R_\lambda^2.$$

□

Proposition II.14. *Let $T \in \mathcal{L}(E)$. If $\lambda_0 \in \rho(T)$, then*

$$D(\lambda_0, \|R_{\lambda_0}\|^{-1}) = \{\lambda \in \mathbb{K}; |\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}\} \subset \rho(T),$$

and for all $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1})$, we have

$$R_\lambda = \sum_{n\geq 0}(-1)^n R_{\lambda_0}^{n+1}(\lambda - \lambda_0)^n.$$

Proof. Let $\lambda_0 \in \rho(T)$. If $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1})$, then

$$\|(\lambda - \lambda_0)R_{\lambda_0}\| < 1.$$

So, the series

$$\sum_{n\geq 0}(-R_{\lambda_0}(\lambda - \lambda_0))^n \in \mathcal{L}(E).$$

This allows us to define the operator S_λ as

$$\begin{aligned} S_\lambda &= R_{\lambda_0}\sum_{n\geq 0}(-R_{\lambda_0}(\lambda - \lambda_0))^n \\ &= \sum_{n\geq 0}(-1)^n(R_{\lambda_0})^{n+1}(\lambda - \lambda_0)^n. \end{aligned}$$

In particular, we have

$$(\lambda_0 I - T)S_\lambda = (R_{\lambda_0})^{-1}S_\lambda = \sum_{n\geq 0}(-1)^n (R_{\lambda_0})^n (\lambda-\lambda_0)^n.$$

So, for all $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1})$, we get

$$\begin{aligned}(\lambda I - T)S_\lambda &= (\lambda_0 I - T + (\lambda-\lambda_0)I)S_\lambda \\ &= (\lambda_0 I - T)S_\lambda + (\lambda-\lambda_0)S_\lambda \\ &= \sum_{n\geq 0}(-1)^n (R_{\lambda_0})^n(\lambda-\lambda_0)^n \\ &- \sum_{n\geq 0}(-1)^{n+1}(R_{\lambda_0})^{n+1}(\lambda-\lambda_0)^{n+1} \\ &= I.\end{aligned}$$

In the same manner, we obtain

$$S_\lambda(\lambda I - T) = I.$$

In conclusion, $D(\lambda_0, \|R_{\lambda_0}\|^{-1}) \subset \rho(T)$ and $R_\lambda = S_\lambda$ for all $\lambda \in D(\lambda_0, \|R_{\lambda_0}\|^{-1})$. □

II.14 Spectrum of an Operator in a Hilbert Space

Proposition II.15. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then, we have*

1. $\ker(T) = (\operatorname{Im} T^*)^\perp$.
2. $\overline{\operatorname{Im} T} = (\ker T^*)^\perp$.

Proof. 1.

$$\begin{aligned}\ker(T) &\overset{\text{def}}{=} \{x \in H;\ \ Tx = 0\} \\ &= \{x \in H,\ \forall y \in H;\ \ \langle Tx, y\rangle = 0\} \\ &= \{x \in H,\ \forall y \in H;\ \ \langle x, T^*y\rangle = 0\} \\ &= (\operatorname{Im} T^*)^\perp.\end{aligned}$$

2. From 1., we have

$$\ker(T^*) = [\operatorname{Im}(T^*)^*]^\perp = (\operatorname{Im} T)^\perp.$$

Therefore,

$$(\ker T^*)^\perp = \left[(\operatorname{Im} T)^\perp\right]^\perp = \overline{\operatorname{Im} T},$$

thus giving the result.

□

Proposition II.16. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then, we have*

1.
$$\rho(T^*) = \left\{\lambda \in \mathbb{K} \mid \overline{\lambda} \in \rho(T)\right\},$$
$$\sigma(T^*) = \left\{\lambda \in \mathbb{K} \mid \overline{\lambda} \in \sigma(T)\right\}.$$

2.
$$\forall \lambda \in \rho(T^*),\ R_\lambda(T^*) = \left(R_{\overline{\lambda}}(T)\right)^*.$$

Proof. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then, we have

1.
$$\begin{aligned} \rho(T^*) &= \{\lambda \in \mathbb{K};\ (\lambda I - T^*)\ \text{ is invertible}\} \\ &= \{\lambda \in \mathbb{K};\ (\overline{\lambda} I - T)\ \text{ is invertible}\} \\ &= \{\lambda \in \mathbb{K} \mid\ \overline{\lambda} \in \rho(T)\}, \end{aligned}$$

$$\begin{aligned} \sigma(T^*) &= (\rho(T^*))^c = \{\lambda \in \mathbb{K};\ \overline{\lambda} \notin \rho(T)\} \\ &= \{\lambda \in \mathbb{K};\ \overline{\lambda} \in \sigma(T)\}. \end{aligned}$$

2. Let $\lambda \in \rho(T^*)$. Then

$$\begin{aligned} R_\lambda(T^*) &= (\lambda I - T^*)^{-1} = \left[(\overline{\lambda} I - T)^*\right]^{-1} \\ &= \left[(\overline{\lambda} I - T)^{-1}\right]^* = \left(R_{\overline{\lambda}}(T)\right)^*, \end{aligned}$$

which is the expected equality.

□

Definition II.20. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then, we have*

1. *We say that T is a normal operator if it commutes with its adjoint, i.e.*
$$T^*T = TT^*.$$

2. *We say that T is a self-adjoint operator if $T = T^*$. (It is also called symmetric when $\mathbb{K} = \mathbb{R}$, and Hermitian when $\mathbb{K} = \mathbb{C}$.)*

Example II.16. *Let $a = (a_n)_{n\geq 0} \in l^\infty(\mathbb{K})$ and T_a the operator defined on $H = l^2(\mathbb{K})$ by*

$$\forall x = (x_n)_{n\geq 0} \in H; \quad T_a(x) = (a_n x_n)_{n\geq 0}.$$

It is clear that $T_a \in \mathcal{L}(H)$, which shows that T_a^* is well-defined. Moreover,

$$\forall x = (x_n)_{n\geq 0} \in H; \quad T_a^*(x) = (\overline{a_n} x_n)_{n\geq 0}.$$

In particular, T_a is self-adjoint if and only if $a \in l^\infty(\mathbb{R})$.

Remark II.7.

1. *If T is self-adjoint, then for all $x \in H$, $\langle Tx, x\rangle \in \mathbb{R}$.*
2. *If $\mathbb{K} = \mathbb{C}$, T is self-adjoint if and only if for all $x \in H$, $\langle Tx, x\rangle \in \mathbb{R}$.*

Proposition II.17 (Eigenvalues of self-adjoint operators)**.** *Let $T \in \mathcal{L}(H)$ be a self-adjoint operator. Then,*

1. *$V_p(T) \subset \mathbb{R}$.*
2. *$\lambda \in V_p(T)$ if and only if $\overline{\mathrm{Im}(\lambda I - T)} \neq H$.*
3. *If $\lambda, \mu \in V_p(T)$ with $\lambda \neq \mu$, then $\ker(\lambda I - T) \perp \ker(\mu I - T)$, i.e., the eigenspaces are pairwise orthogonal.*

Proof.

1. If $\lambda \in V_p(T)$, then there exists $x \in H \setminus \{0\}$ such that $\lambda x = Tx$. Thus,
$$\lambda = \frac{\langle \lambda x, x\rangle}{\|x\|^2} = \frac{\langle Tx, x\rangle}{\|x\|^2} \in \mathbb{R}.$$
2. According to Proposition (II.15), we have
$$\overline{\mathrm{Im}(\lambda I - T)} = \big(\ker(\lambda I - T)^*\big)^\perp = \big(\ker(\overline{\lambda} I - T^*)\big)^\perp.$$
Since $V_p(T) \subset \mathbb{R}$, we have
$$\begin{aligned} \lambda \in V_p(T) &\iff \overline{\lambda} \in V_p(T) \iff \ker(\overline{\lambda} I - T^*) \neq \{0\} \\ &\iff \big[\ker(\overline{\lambda} I - T^*)\big] \neq H = \overline{\mathrm{Im}(\lambda I - T)}. \end{aligned}$$
3. If $x \in \ker(\lambda I - T)$ and $y \in \ker(\mu I - T)$, then $Tx = \lambda x$ and $Ty = \mu y$. Since $T = T^*$ and $\mu \in \mathbb{R}$, we obtain
$$\begin{aligned} (\lambda - \mu)\langle x, y\rangle &= \langle \lambda x, y\rangle - \langle x, \mu y\rangle \\ &= \langle Tx, y\rangle - \langle x, Ty\rangle \\ &= 0, \end{aligned}$$
which implies $\langle x, y\rangle = 0$ since $\lambda \neq \mu$.
In other words, $\ker(\lambda I - T) \perp \ker(\mu I - T)$.

□

II.15 Exercices

Exercise 1

Let E be a normed vector space, and let $T \in \mathcal{L}(E)$ be a compact linear operator. We denote by I the identity operator on E.

1. For $\alpha \in \mathbb{C}$, $\alpha \neq 0$, define the subspace:

$$N_\alpha = \ker(T - \alpha I) = \{x \in E \,;\, T(x) = \alpha x\}.$$

   Prove that N_α is finite-dimensional using the Riesz theorem. (Recall: B_E compact $\Longleftrightarrow \dim E < \infty$).

2. Let l_2 be the space of square-summable complex sequences. Consider the linear operator

$$S : l_2 \longrightarrow l_2$$

   defined by

$$S(x) = (x_1, 0, x_3, 0, x_5, \dots).$$

   Prove that S is not compact.

Exercise 2

Let E be a vector space equipped with two norms $||.||_1$ and $||.||_2$. Assume that E is a Banach space with respect to both norms.
Show that if

$$||.||_2 \leq k||.||_1,$$

then the two norms are equivalent.

Exercise 3

Let U and V be two Banach spaces, and let $T : U \longrightarrow V$ be a linear operator.

1. Define the norm

$$||u||_T = ||Tu||_V + ||u||_U.$$

   Show that $(U, ||.||_T)$ is a normed space and a Banach space.

2. Prove that T is continuous if and only if the graph of T is closed.

II.16 Solved problem

Let $T : C_{00} \to C_{00}$ be defined by:

$$T(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots\right),$$

where $x = (x_n) \in C_{00}$, the space of sequences with finitely many nonzero terms, equipped with the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

1. Show that T is continuous and calculate $\|T\|$.

2. Show that T is bijective but T^{-1} is not continuous.

3. What can you conclude.

Solution

1. Show that T is linear, continuous and calculate $\|T\|$. Firstly, it's clear that T is linear.

   Now, we need to show that T is continuous and calculate its norm.

   The operator T is defined by

   $$T(x) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right).$$

   The norm on C_{00} is given by

   $$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

   We have

   $$\|T(x)\|_\infty = \sup_{n\in\mathbb{N}} \left|\frac{x_n}{n}\right| = \sup_{n\in\mathbb{N}} \frac{|x_n|}{n}.$$

   Since $x \in C_{00}$, there exists an index N such that for all $n > N$, $x_n = 0$. Thus

   $$\|T(x)\|_\infty = \sup_{n\in\mathbb{N}} \frac{|x_n|}{n} \leq \sup_{n\in\mathbb{N}} \frac{\|x\|_\infty}{n} = \|x\|_\infty.$$

   Thus, T is continuous, and we have $\|T\| \leq 1$. On the other hand, let $e_1 = (1, 0, 0, ...)$ be a vector. Thus $e_1 \in C_{00}$, $\|e_1\|_\infty = 1$, $T(e_1) = e_1$ and $\|T(e_1)\|_\infty = 1$. So,

   $$\|T\| = \sup_{x\in C_{00}, \|x\|_\infty = 1} \|T(x)\|_\infty. \geq \|T(e_1)\|_\infty = 1.$$

2. Show that T is bijective

   - Injectivity of T: since $T(x) = 0 \Rightarrow x = 0$.
   - Surjectivity of T: indeed if $y = (y_1, y_2, ...) \in C_{00}$.
     Let $u = (y_1, 2y_2, 3y_3...) \in C_{00}$. Thus, $T(u) = y$. Thus, T^{-1} is defined by $T^{-1}(y) = (ny_n)$ for $y = (y_n)_{n\in\mathbb{N}^*} \in C_{00}$.
   - T^{-1} is not continuous. We consider, for $n \in \mathbb{N}^*$, the vector $v_n = (1, 1, ..., 1, 0, 0, ...)$, Thus $\|v_n\|_\infty = 1$, $T^{-1}(v_n) = (1, 2, 3, ...n, 0, 0, ...)$ and $\|T^{-1}(v_n)\|_\infty = n$. Also,

     $$\sup_{y\in C_{00}, \|y\|_\infty = 1} \|T^{-1}(y)\|_\infty. \geq \|T^{-1}(v_n)\|_\infty = n \text{ for all } n \in \mathbb{N}^*.$$

     Which means that T^{-1} is not continuous.

3. Conclusion: We have a linear, continuous, and bijective operator on C_{00}, but its inverse is not continuous. We deduce, using the Banach Isomorphism Theorem, that C_{00} is not a Banach space.

Chapter III

Fundamental theorems of functional analysis

Contents

III.1 The Baire category Theorem

Theorem III.1. *[3] Let X be a complete metric space, and let $(X_n)_{n\geq 1}$ be a sequence of closed subsets of X. Suppose that*

$$int(X_n) = \emptyset \quad \textit{for all } n \geq 1.$$

Then,

$$int\left(\bigcup_{n=1}^{\infty} X_n\right) = \emptyset.$$

In other words, in a complete metric space, a countable union of closed sets with empty interior also has an empty interior.

Proof. Let $O_n = (X_n)^C$. Then, O_n is open and dense, and we need to show that $G = \bigcap_{n=1}^{\infty} O_n$ is dense in X. Let U be a nonempty open subset of X. We will prove that U intersects G at some point x, which is the limit of a sequence that we now define.

Step 0. Let $x_0 \in U$. Since U is open, there exists $r_0 > 0$ such that $B(x_0, r_0) \subseteq U$, where $B(x_0, r_0)$ is the open ball centered at x_0 with radius r_0.

Step 1. Since O_1 is dense, it intersects $B(x_0, r_0)$ at a point x_1. Since $O_1 \cap B(x_0, r_0)$ is open, there exists $r_1 > 0$ such that $B(x_1, r_1) \subseteq O_1 \cap B(x_0, r_0)$. We can choose r_1 such that $r_1 < \frac{r_0}{2}$.

Step 2. Since O_2 is dense, it intersects $B(x_1, r_1)$ at a point x_2. Since $O_2 \cap B(x_1, r_1)$ is open, there exists $r_2 > 0$ such that $B(x_2, r_2) \subseteq O_2 \cap B(x_1, r_1)$. We can choose r_2 such that $r_2 < \frac{r_1}{2}$.

Continuing in this way, we construct two sequences (x_n) and (r_n) such that

$$B(x_{n+1}, r_{n+1}) \subseteq O_{n+1} \cap B(x_n, r_n) \quad \text{for all} \quad n \geq 0$$

and

$$r_{n+1} < \frac{r_n}{2}.$$

It follows that $x_{n+p} \in B(x_n, r_n)$, i.e., $d(x_{n+p}, x_n) < r_n \to 0$ as $n \to \infty$. This means that (x_n) is a Cauchy sequence. Since the space is complete, (x_n) converges to a limit x.

But since $x_{n+p} \in B(x_n, r_n)$ for all $n \geq 0$ and all $p \geq 0$, by letting $p \to \infty$, we get

$$x \in B(x_n, r_n) \quad \text{for all} \quad n \geq 0.$$

By construction, $B(x_n, r_n) \subseteq O_n$ and $B(x_n, r_n) \subseteq B(x_0, r_0) \subseteq U$. Therefore, $x \in U \cap G$. □

Remark III.1. *This theorem is often used in the following form. Let X be a complete metric space and (X_n) a sequence of closed subsets such that*

$$\bigcup_{n=1}^{\infty} X_n = X.$$

Then, there exists a positive integer m such that $\overset{\circ}{X_m} \neq \emptyset$.

III.2 The Uniform Boundedness Principle

Theorem III.2. *[9][The Uniform Boundedness Principle] Let E be a Banach space, F a normed space over the same field $\mathbb{K}$, and $\mathcal{A} \subset B(E, F)$ a non-empty set of bounded linear operators from E to F, such that*

$$\{A(x) \mid \text{for all } A \in \mathcal{A}\} \subset F$$

is bounded for every $x \in E$. Then the set $\mathcal{A}$ is bounded in $B(E, F)$.

Proof. Let $F_n = \{x \in E \mid \|A(x)\| \leq n, \ A \in \mathcal{A}\}$, $n = 1, 2, \ldots$.
Then, F_n is a closed set in E because if $\{x_m\}$ is a Cauchy sequence in F_n, thus $\|x_m - x\|_E \to 0$ (since E is complete).
For all $A \in \mathcal{A}$, we have $Ax_m \to Ax$, which gives $\|Ax_m\| \to \|Ax\|$ (the continuity of the norm).
On the other hand,

$$\|Ax\| = \lim_{m} \|Ax_m\| \leq \lim_{m \to +\infty} n = n.$$

So, $x \in F_n$ and F_n is closed.

- We now show that $E = \bigcup_{i=1}^{\infty} F_n$.

Consider $x_0 \in E$ and $\sup\{\|Ax_0\|, \ A \in \mathcal{A}\} = M_0$.
Let n_0 be the first integer greater than or equal to M_0, for all $A \in \mathcal{A}$, $\|Ax_0\| \leq n_0$, i.e. $x_0 \in F_{n_0} \subset \bigcup_{i=1}^{\infty} F_n$, so $E = \bigcup_{i=1}^{\infty} F_n$.
Since E is a Banach space and by the Baire Category Theorem, there exists $N \in \mathbb{N}$ such that $\overset{\frown}{F_N}^{\circ} \neq \emptyset$.
This implies the existence of $x_N \in \overset{\frown}{F_N}^{\circ}$ and $B(x_N, \rho) \subset F_N$, so $B_f(x_N, \rho) \subset F_N$.
For $w \in E$, with $\|w\| \leq \rho$, we have $w + x_N \in B_f(x_N, \rho) \subset F_N$.

$$\begin{aligned}\forall A \in \mathcal{A},\ \|A(w)\| &= \|A(w+x_N) - A(x_N)\| \\ &\le \|A(w+x_N)\| + \|A(x_N)\| \\ &\le N + N = 2N.\end{aligned}$$

For $x \in E$, we set $w = \rho\dfrac{x}{\|x\|}$. Then,

$$\forall A \in \mathcal{A},\ \forall x \in E,\ \|A\left(\rho\frac{x}{\|x\|}\right)\| \le 2N.$$

So,

$$\forall A \in \mathcal{A},\ \forall x \in E,\ \|A(x)\| \le \frac{2N}{\rho}\|x\|.$$

Hence, $\forall A \in \mathcal{A},\ \|A\| \le \frac{2N}{\rho}$ and $\mathcal{A}$ is bounded. □

Theorem III.3. *[9] [Banach-Steinhaus theorem] Let E be a Banach space, F a normed space, and let $\{A_n\}_n \subset B(E,F)$ be a sequence of bounded linear operators. Suppose that $\{A_n(x)\}_n \subset F$ converges to $A(x) \subset F$ for all $x \in E$, then $A : E \longrightarrow F$ is a continuous linear operator.*

Proof. For all $x \in E$, we have

$$\begin{aligned}A(\alpha x + \beta y) &= \lim_n A_n(\alpha x + \beta y) \\ &= \alpha \lim_n A_n(x) + \beta \lim_n A_n(y) \\ &= \alpha A(x) + \beta A(y).\end{aligned}$$

Thus, A is linear. Since, for all $x \in E$, $\lim_n \|A_n(x)\| = \|A(x)\|$ implies that $\|A_n(x)\| \in \mathbb{R}$ is bounded for all $n \in \mathbb{N}$. That is, for all $x \in E$, $\{\|A_n(x)\|, n \in \mathbb{N}\}$ is bounded. By the Uniform Boundedness Principle, $\|A_n\|$ is bounded for all $n \in \mathbb{N}$. Therefore, $\|A(x)\| = \lim_n \|A_n(x)\| = \liminf_n \|A_n(x)\| \le \liminf_n \|A_n\|\|x\|$. Hence, $\forall x \in E,\ \|A(x)\| \le K\|x\|$. □

III.3 Analytic Form of the Hahn-Banach Theorem

Definition III.1. *We say that f is a convex or sublinear functional if f is a mapping from E to $\mathbb{K}$ satisfying:*

- $f(x+y) \leq f(x) + f(y)$ *(subadditive),*
- $f(\alpha x) = \alpha f(x)$ *(positively homogeneous),*

for all x, y and for all $\alpha \geq 0$.

Remark III.2. *A sublinear functional f on the space E has the following properties:*

$$f(0) = 0 \text{ and } -f(-x) \leq f(x).$$

Indeed,

$$f(0 \cdot x) = 0 \cdot f(x)$$

and

$$0 = f(x + (-x)) \leq f(x) + f(-x),$$

from which we deduce that $-f(-x) \leq f(x)$.

Example III.1. *Let $(E, \|.\|)$ be a normed space. Then, the norm $\|.\|$ is a convex functional on E.*

Lemma III.1. *Let E be a linear space over $\mathbb{R}$, M a proper subspace of E (i.e., $M \subsetneq E$), p a convex functional on E, and $f \in L(M, \mathbb{R})$ satisfying*

$$f(x) \leq p(x), \quad \forall x \in M.$$

Fix $x_0 \in E \setminus M$ and set $M_1 = M \oplus [x_0]$. Then there exists an extension $F \in L(M_1, \mathbb{R})$ of f such that

$$F(x) \leq p(x), \quad \text{for all } x \in M_1.$$

Proof. For $m_1, m_2 \in M$, we have

$$\begin{aligned} f(m_1) - f(m_2) &\leq p(m_1 - m_2) \\ &\leq p(m_1 - x_0 + x_0 - m_2) \\ &\leq p(m_1 + x_0) + p(-(x_0 + m_2)). \end{aligned}$$

Thus,

$$-f(m_2) - p(-(x_0 + m_2)) \leq p(m_1 + x_0) - f(m_1)$$

and

$$\alpha = \sup_{m_2 \in M} \big(- f(m_2) - p(-(x_0 + m_2))\big) \leq \beta = \inf_{m_1 \in M} \big(p(m_1 + x_0) - f(m_1)\big).$$

Let us choose $\gamma \in \mathbb{R}$, such that

$$\alpha \leq \gamma \leq \beta.$$

Then for all $z \in M$, we have

$$(-f(z) - p(-(x_0 + z)) \leq \gamma \leq p(z + x_0) - f(z). \tag{III.3.1}$$

If $x \in M_1 = M \oplus [x_0]$, then x can be written uniquely as

$$x = m + \lambda x_0,\ m \in M \text{ and } \lambda \in \mathbb{R}.$$

Now, we define the functional $F : M_1 \longrightarrow \mathbb{R}$ by

$$F(x) = f(m) + \lambda\gamma.$$

- If $\lambda = 0$,

  $$F(x) = F(m + 0 \cdot x_0) = f(m) \leq p(m),$$

  so $F|_M = f$.

- If $\lambda > 0$.
  For all $x \in E$, let $z = \dfrac{m}{\lambda}$, then the right side of (III.3.1) becomes

  $$\gamma \leq p\left(\frac{m}{\lambda} + x_0\right) - f\left(\frac{m}{\lambda}\right). \tag{III.3.2}$$

  By multiplying (III.3.2) by $\lambda > 0$, we get

  $$\begin{array}{rcl} \lambda\gamma & \leq & p(m + x_0) \\ f(m) + \lambda\gamma & \leq & p(m + x_0), \end{array}$$

  which implies $F(x) \leq p(x)$.

- If $\lambda < 0$. From the first term in (III.3.1) and for $z = \dfrac{m}{\lambda}$, we have

$$\gamma \geq -p\left(-\frac{m}{\lambda} - x_0\right) - f\left(\frac{m}{\lambda}\right). \tag{III.3.3}$$

  Multiplying (III.3.3) by $\lambda < 0$, we obtain

$$-\lambda p\left(-\frac{m}{\lambda} - x_0\right) - \lambda f\left(\frac{m}{\lambda}\right) \geq \lambda\gamma, \tag{III.3.4}$$

  which implies that $F(x) \leq p(x)$.

□

Theorem III.4. *[1][Hahn-Banach] Let E be a vector space over the field $\mathbb{R}$ and M a subspace of E. Let p be a convex functional on E and $f \in L(E, \mathbb{R})$ such that*

$$f(x) \leq p(x) \quad \forall x \in M.$$

Then there exists $F \in L(E, \mathbb{R})$ such that

(i) $F|_M = f$,

(ii) $F(x) \leq p(x)$ *for all* $x \in E$.

As a consequence of the Hahn-Banach theorem, we have the following result.

Theorem III.5. *Let E be a real vector space, $x_0 \in E$ a fixed element, and $p : E \to \mathbb{R}_+$ a sublinear functional. Then, there exists $f \in L(E, \mathbb{R})$ such that:*

(i) $f(x_0) = p(x_0)$,

(ii) $f(x) \leq p(x)$ *for all* $x \in E$.

Proof.

- If $x_0 = 0$, we take $f = 0$.

- Let $x_0 \neq 0$ and $M = [x_0] = \{\lambda x_0 : \lambda \in \mathbb{R}\}$.
  Define $f_0 : M \to \mathbb{R}$ by

$$f_0(\lambda x_0) = \lambda p(x_0) \quad \text{for all } x = \lambda x_0 \in M.$$

  Then, $f_0 \in L(E, \mathbb{R})$ and $f(x_0) = p(x_0)$.
  Now, we show (ii).
  Let $x \in M$, i.e., $x = \lambda x_0$. Then,

- If $\lambda \geq 0$, we have

$$f_0(\lambda x_0) = \lambda p(x_0) = p(x).$$

- If $\lambda < 0$, we have

$$\begin{aligned} f_0(x) = f_0(\lambda x_0) &= \lambda p(x_0) \\ &= -\lambda p(x_0) = -p(-\lambda x_0) \\ &\leq p(\lambda x_0). \end{aligned}$$

The Hahn-Banach theorem implies the existence of $f \in L(E, \mathbb{R})$ such that

$$\begin{aligned} f|_M = f_0 \quad &\text{i.e., } f(x_0) = f_0(x_0) = p(x_0), \\ f(x) \leq p(x) \quad &\text{for all } x \in E. \end{aligned}$$

□

Theorem III.6. *[9] Let E be a normed vector space, M a subspace of E, and f a bounded linear functional on M. Then, there exists a bounded linear extension F of f to E such that $\|F\| = \|f\|$.*

Proof. Let $(E, \|\cdot\|)$ be a normed vector space, M a subspace of E, and $f : M \to \mathbb{R}$ a bounded linear functional. Define a sublinear functional $p : E \to \mathbb{R}$ by $p(x) = \|f\|\|x\|$. Since $|f(x)| \leq \|f\|\|x\|$ for all $x \in M$, we have $f(x) \leq p(x)$ for all $x \in M$. By Theorem III.5, there exists a bounded linear extension $F : E \to \mathbb{R}$ of f such that $F(x) \leq p(x)$ for all $x \in E$. Since $\forall x \in E, -F(x) = F(-x) \leq p(-x) = p(x)$. This implies that $\forall x \in E;\ |F(x)| \leq p(x)$ and $\|F\| \leq \|f\|$.
On the other hand,

$$\|F\| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |F(x)| \geq \sup_{\substack{\|x\| \leq 1 \\ x \in M}} |f(x)| = \|f\|.$$

Hence, $\|F\| = \|f\|$. □

Example III.2. *Let $X = \mathbb{R}^2$ equipped with the infinity norm $\|\cdot\|$, defined as*

$$\|(x_1, x_2)\| = \max(|x_1|, |x_2|).$$

Consider the subspace $Y = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$, *which corresponds to the first coordinate of* $\mathbb{R}^2$. *Define a linear functional* $f : Y \to \mathbb{R}$ *as*

$$f(x_1, 0) = 3x_1, \quad \text{for all } x_1 \in \mathbb{R}.$$

We can verify that f *is bounded on* Y *with respect to the infinity norm, since*

$$|f(x_1, 0)| = |3x_1| = 3|x_1| \leq 3\|(x_1, 0)\|.$$

Thus, f *is a bounded linear functional on* Y *with norm* $\|f\| = 3$.
Our goal is to extend f *to a functional* $F : X \to \mathbb{R}$ *defined on all of* X *such that* F *remains linear and* $\|F\| = 3$. *In other words, we seek* F *such that*

$$F(x_1, x_2) = f(x_1, 0) = 3x_1 \quad \text{for all } (x_1, 0) \in Y$$

and

$$|F(x_1, x_2)| \leq 3\|(x_1, x_2)\| \quad \text{for all } (x_1, x_2) \in X.$$

Solution using the Hahn-Banach Theorem *The Hahn-Banach Theorem guarantees that such an extension* F *exists with the same norm,* $\|F\| = 3$.
We define F *as follows*

$$F(x_1, x_2) = 3x_1.$$

To verify that this extension satisfies the required conditions:

1. *Compatibility with* f*: For all* $(x_1, 0) \in Y$, *we have*

   $$F(x_1, 0) = 3x_1 = f(x_1, 0).$$

   So F *extends* f *on* Y.

2. *Boundedness: We need to verify that* $|F(x_1, x_2)| \leq 3\|(x_1, x_2)\|$ *for all* $(x_1, x_2) \in X$. *Indeed,*

   $$|F(x_1, x_2)| = |3x_1| \leq 3\max(|x_1|, |x_2|) = 3\|(x_1, x_2)\|.$$

This shows that $\|F\| \leq 3$. *Thus,* $F(x_1, x_2) = 3x_1$ *is an extension of* f *to* X *that satisfies the conditions of the Hahn-Banach Theorem. Therefore,* F *is the desired functional.*

As an important consequence of the Hahn-Banach Theorem, we state the following result.

Corollary III.1. *Let* E *be a normed space,* $x_0 \in E \setminus \{0\}$, *and* $\mu_0 > 0$. *Then, there exists a linear and continuous functional on* E *such that*

$$F(x_0) = \mu_0\|x_0\| \quad \text{and} \quad \|F\| = \mu_0.$$

Proof. Let M be the subspace spanned by x_0, i.e., $M = [x_0]$.
We define the functional $f(x) = f(\lambda x_0) = \mu_0 \lambda \|x_0\|$, where $x \in M$ and $\lambda \in \mathbb{K}$.
Obviously, $f(x_0) = \mu_0 \|x_0\|$ and

$$\|f\| = \sup_{\substack{x \neq 0 \\ x \in M}} \frac{|f(x)|}{\|x\|} = \sup_{\lambda \neq 0} \frac{\mu_0 \lambda \|x_0\|}{\lambda \|x_0\|} = \mu_0.$$

According to Theorem III.6, there exists a unique extension F of the functional f such that

$$F(x_0) = \mu_0 \|x_0\| \quad \text{and} \quad \|F\| = \mu_0.$$

□

III.4 Geometric Form of the Hahn-Banach Theorem: Separation of Convex Sets

Definition III.2. *A hyperplane is a set of the form*

$$H = \{x \in E : f(x) = \alpha\},$$

where f is a linear form on E, not identically zero, and $\alpha \in \mathbb{R}$. We say that H is a hyperplane with the equation $[f = \alpha]$.

Proposition III.1. *[1] The hyperplane with equation $[f = \alpha]$ is closed if and only if f is continuous.*

Proof. It is clear that if f is continuous, H is closed.
Conversely, suppose that CH, the complement of H, is open and non-empty.
Let $x_0 \in CH$, and assume, for example, that $f(x_0) < \alpha$.
Let $r > 0$ such that $B(x_0, r) \subset CH$. We have

$$f(x) < \alpha, \quad \text{for all} \quad x \in B(x_0, r). \tag{III.4.1}$$

Indeed, suppose that $f(x_1) > \alpha$ for some $x_1 \in B(x_0, r)$.
The line segment: $[x_0, x_1] = \{x_t = (1-t)x_0 + t x_1, \quad t \in [0,1]\} \subset B(x_0, r)$, thus $f(x_t) \neq \alpha \quad \forall t \in [0,1]$. Moreover, $f(x_t) = \alpha$, for $t = \dfrac{f(x_1) - \alpha}{f(x_1) - f(x_0)}$, which is absurd, so (III.4.1) is proven. It follows from (III.4.1) that

$$f(x_0 + rz) < \alpha, \quad \forall z \in B(0,1).$$

Consequently, f is continuous and $\|f\| \leq \dfrac{1}{r}(\alpha - f(x_0))$. □

III.4. Geometric Form of the Hahn-Banach Theorem: Separation of Convex Sets

Definition III.3. *Let $A \subset E$ and $B \subset E$. We say that the hyperplane with equation $[f = \alpha]$ separates A and B if*

$$f(x) \leq \alpha \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha \quad \forall x \in B.$$

We say that the hyperplane with equation $[f = \alpha]$ separates A and B in the strict sense if there exists $\varepsilon > 0$ such that

$$f(x) \leq \alpha - \varepsilon \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha + \varepsilon \quad \forall x \in B.$$

Theorem III.7. *[3][Geometric Form of the Hahn-Banach Theorem: Separation of Convex Sets] Let A and B be disjoint, non-empty, convex subsets of a topological vector space E, with A open. Then, there exists a closed hyperplane separating A and B.*

For the proof of this theorem, we shall require the following two lemmas:

Lemma III.2. *[Gauge of a convex set]*
Let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$, define

$$p(x) = \inf \left\{ \alpha > 0, \;\; \alpha^{-1}x \in C \right\}.$$

We say that p is the Gauge of C.
Then p satisfies the following properties:

1. $p(tx) = tp(x) \;\; t > 0 \;\; x \in E$.
2. $p(x) < 1 \iff x \in C$.
3. $p(x + y) \leq p(x) + p(y) \; x, y \in E$.
4. $0 \leq p(x) \leq M\|x\|, \; \forall x \in E$.

Proof. 1. Obvious.

2. Suppose first that $x \in C$. Since C is open, $(1 + \varepsilon)x \in C$ for sufficiently small ε. Thus, $p(x) \leq \frac{1}{1+\varepsilon} < 1$.
Reciprocally, if $p(x) < 1$, there exists $0 < \alpha < 1$ such that $\alpha^{-1}x \in C$, so $x = \alpha(\alpha^{-1}x) + (1 - \alpha)0 \in C$.

3. Let $x, y \in E$ and $\varepsilon > 0$. By (1) and (2), we have $\frac{x}{p(x)+\varepsilon} \in C$ and $\frac{y}{p(y)+\varepsilon} \in C$, so,

$$\frac{tx}{p(x) + \varepsilon} + \frac{(1 - t)y}{p(y) + \varepsilon} \in C \quad \text{for all} \quad t \in [0, 1].$$

In particular, for $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$, we obtain $\frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$.
It follows, thanks to (1) and (2), that

$$p(x+y) \leq p(x) + p(y) + 2\varepsilon \quad \forall \varepsilon > 0,$$

which gives (3).

4. Let $r > 0$ such that $B(0, r) \subset C$.
Since $r\frac{x}{\|x\|} \in C$, we have $p(x) \leq \frac{1}{r}\|x\|$, for all $x \in E$.

□

Lemma III.3. *Let C be a non-empty open convex set, and let $x_0 \in E$ with $x_0 \notin C$. Then, there exists $f \in E'$ such that*

$$f(x) < f(x_0) \quad \textit{for all} \quad x \in C.$$

Proof. Suppose that $0 \in C$ and introduce the gauge of C, denoted by p. Consider $G = \mathbb{R}x_0$ and the linear functional g defined on G by $g(tx_0) = t$ for all $t \in \mathbb{R}$. It is clear that $g(x) \leq p(x)$ for all $x \in G$. By the Hahn-Banach theorem, there exists a linear functional f extending g to E such that: $f(x) \leq p(x)$, for all $x \in E$. In particular, $f(x_0) = 1$ and f is continuous thanks to (4). On the other hand, it follows from (2) that $f(x) < 1$, for all $x \in C$. □

Proof. **of Theorem (III.7):** Let $C = A - B$, so that C is a convex and open set (note that $C = \cup_{y \in B}(A - y)$ and $0 \notin C$ since $A \cap B = \varnothing$).
According to Lemma (III.3), there exists $f \in E'$ such that $f(z) < 0$, for all $z \in C$. This means $f(x) < f(y)$ for all $x \in A$, $y \in B$.
Now, fix $\alpha \in \mathbb{R}$ such that $\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$, and thus, the hyperplane defined by the equation $[f = \alpha]$ separates A and B. □

Theorem III.8. *[1][Banach, Second Geometric Form] Let $A \subset E$ and $B \subset E$ be two non-empty, convex, and disjoint sets. Assume that A is closed and B is compact. Then, there exists a closed hyperplane that strictly separates A and B.*
In other words, there exists a continuous linear functional $f \in E'$ and constants $\alpha, \beta \in \mathbb{R}$ such that

$$f(x) \leq \alpha \quad \textit{for all} \quad x \in A \quad \textit{and} \quad f(y) \geq \beta \quad \textit{for all} \quad y \in B,$$

with $\alpha < \beta$.

Proof. Let $C = A - B$. First, C is convex. Second, C is closed. Indeed, let $c \in C$. Then, there exist two sequences (x_n) and (y_n) in A and B, respectively, such that $x_n - y_n \to c$. Since B is compact, the sequence (y_n) has a subsequence that converges to some $y \in B$. Without loss of generality, we denote this subsequence by (y_n).
Now, the sequence (x_n) converges to some $x := c + y$. Since A is closed, we have $x \in A$. It follows that $c = x - y \in A - B = C$. Third, $0 \notin C$ because $A \cap B = \varnothing$. Therefore, there exists $r > 0$ such that $B(0, r) \cap C = \varnothing$. By the previous theorem, there exists $f \in E'$ with $f \neq 0$ such that

$$f(x - y) \leq f(rz), \quad \forall x \in A,\ \forall y \in B,\ \forall z \in B(0, 1).$$

(This implies that f separates $rB(0, 1)$ and C.)
This means that

$$f(x - y) \leq rf(z), \quad \forall z \in B_E.$$

Recall that $\inf_{z \in B_E} f(z) = -\|f\|$. It follows that

$$f(x - y) \leq r \inf_{z \in B_E} f(z) = -r\|f\|, \quad \forall x \in A,\ \forall y \in B.$$

Let $\varepsilon = \frac{1}{2}r\|f\|$. Then

$$f(x) + \varepsilon \leq f(y) - \varepsilon, \quad \forall x \in A,\ \forall y \in B.$$

Choosing β such that

$$\sup_{x \in A} f(x) + \varepsilon \leq \beta \leq \inf_{y \in B} f(y) - \varepsilon,$$

we see that the hyperplane $[f = \beta]$ strictly separates A and B. □

Corollary III.2. *Let $F \subset E$ be a vector subspace such that $\overline{F} \neq E$. Then, there exists $f \in E'$ with $f \neq 0$ such that*

$$\langle f, x \rangle = 0, \quad \forall x \in F.$$

Proof. Let $x_0 \in E$ and $x_0 \notin \overline{F}$. We apply Theorem (III.8) with $A = \overline{F}$ and $B = \{x_0\}$. There exists a linear functional $f \in E'$ such that the hyperplane with equation $[f = \alpha]$ strictly separates $\overline{F}$ and $\{x_0\}$. We have

$$\langle f, x \rangle < \alpha < \langle f, x_0 \rangle, \quad x \in F.$$

Thus, it follows that $\langle f, x \rangle = 0, \quad$ for all $x \in F$.
Since, for all $x \in F$, we have $\lambda\langle f, x \rangle < \alpha$ for every $\lambda \in \mathbb{R}$, ($\mathbb{R}$ will be bounded). □

Reflexivity of Closed Subspaces

Theorem III.9. *Every closed subspace of a reflexive space is reflexive.*

Proof. We recall that a Banach space X is reflexive if the map $X \to X^{**}$ given by $x \mapsto (x^* \mapsto \langle x^*, x\rangle)$ is surjective.
Proof. We want to show that for any $y^{**} \in Y^{**}$, there exists $y_0 \in Y$ such that

$$\langle y^{**}, y^*\rangle = \langle y^*, y_0\rangle \quad \text{for all } y^* \in Y^*.$$

To prove this, given y^{**}, define $x^{**} \in X^{**}$ by

$$\langle x^{**}, x^*\rangle = \langle y^{**}, x^*|_Y\rangle.$$

By the reflexivity of X, there exists $x_0 \in X$ such that

$$\langle y^{**}, x^*|_Y\rangle = \langle x^{**}, x^*\rangle = \langle x^*, x_0\rangle \tag{1}$$

for any $x^* \in X^*$. It remains to show that $x_0 \in Y$.
Assume to the contrary that $x_0 \notin Y$. Then apply the Hahn-Banach Theorem (noting that Y is a closed subspace of X) to get an $x_0^* \in X^*$ such that $x_0^*|_Y \equiv 0$ and $\langle x_0^*, x_0\rangle > 0$. Substituting this x_0^* into (1), we see that

$$0 = \langle y^{**}, x_0^*|_Y\rangle = \langle x^{**}, x_0^*\rangle = \langle x_0^*, x_0\rangle > 0,$$

which is a contradiction. Therefore, $x_0 \in Y$, and the proof is complete.

□

III.5 Exercises

Exercise 1 Show that, for every $x_0 \in E$, there exists $f_0 \in E'$ such that

$$\|f_0\| = \|x_0\| \quad \text{and} \quad f_0(x_0) = \|x_0\|^2.$$

Exercise 2 Let $(X, \|.\|)$ be a normed space and let $X_0 \subset X$ be a linear subspace of X such that $\overline{X_0} \neq X$. Let $x_0 \in X - \overline{X_0}$ be a fixed element and

$$d = \text{dis}(x_0, \overline{X_0}) = \inf_{y \in \overline{X_0}} \|x_0 - y\|.$$

Show that there exists a linear functional f such that

$$f|_{\overline{X_0}} = 0, \quad \|f\| = 1, \quad \text{and} \quad f(x_0) = d.$$

Exercise 3

1. Show that if B is a subset of a Banach space (say, complex) E such that

$$\sup_{x \in B} |\varphi(x)| < +\infty$$

for every $\varphi \in E^*$, then B is a bounded subset of E. (Hint: Consider the continuous linear functionals $\hat{x} \in E^{**}$ associated with elements $x \in B$).

2. Let $U = (u_n)_{n \geq 1}$ be a sequence of complex numbers. Assume that for every sequence $a = (a_n)_{n \geq 1}$ such that $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$, the series $\sum_{n=1}^{\infty} a_n u_n$ converges. Show that

$$\sum_{n=1}^{\infty} |u_n|^2 < +\infty.$$

(Hint: Prove that the partial sums of the series are uniformly bounded.)

Exercise 4
Let E be a vector space, and let f, g be linear functionals from E to $\mathbb{R}$. Assume that

$$\ker f \subset \ker g.$$

We want to show that $g = \lambda f$.

1. Consider the map $\varphi : E \longrightarrow \mathbb{R}^2$ defined by:

$$\varphi(x) = (f(x), g(x)).$$

Prove that $(0, 1) \notin \text{Im}(\varphi)$.

2. Separate $(0,1)$ and $\mathrm{Im}(\varphi)$ by a hyperplane in $\mathbb{R}^2$.

3. Conclude the result.

Exercise 5 Let X be a real normed vector space, and let $C \subseteq X$ be a closed convex set.

1. Let $(\varphi_i)_{i\in I}$ be a family of affine functions on C with real values, such that
$$\sup_{x\in C} \varphi_i(x) < \infty \quad \text{for all } i \in I.$$
Prove that the function $u = \sup_{i\in I} \varphi_i$ is convex.

2. Let $u : C \longrightarrow \mathbb{R}$ be a continuous convex function.

   (a) Show that the set
   $$\varphi = \{(x,a) \in C \times \mathbb{R} : u(x) \leq a\}$$
   is a closed convex subset of $C \times \mathbb{R}$.

   (b) Let $x_0 \in C$. Prove that for every $r < u(x_0)$, there exist $x^* \in X^*$ and $\lambda \in \mathbb{R}$ such that
   $$x^*(x) + \lambda a < x^*(x_0) + \lambda r \quad \text{for all } (x,a) \in C \times \mathbb{R}.$$

III.6 Solved problem

We aim, for $1 < p < \infty$, to find the dual of ℓ^p. Let $x = (x_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that the series

$$\sum_{n=0}^{+\infty} x_n y_n$$

is convergent for every $y = (y_n)_{n\in\mathbb{N}} \in \ell^p$, $(1 < p < \infty)$.

1. For an integer $N \geq 0$, compute the norm of the linear functional $U_N : \ell^p \to \mathbb{C}$ defined by

$$U_N(y) = \sum_{n=0}^{N} x_n y_n, \quad \forall y \in \ell^p.$$

   (Use the Minkowski inequality.)

2. Deduce that $x \in \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. (Use the Banach-Steinhaus Theorem.)

3. U is defined on ℓ^p by

$$U(y) = \sum_{n=0}^{+\infty} x_n y_n.$$

   Find the limit of $\|U - U_N\|$ as $N \to \infty$.

4. Identify the dual of ℓ^p.

5. Identify the dual of ℓ^1.

6. Identify the dual of $c_0 = \{x = (x_n)_n : x_n \to 0\}$ with $\|x\| = \sup_{n\in\mathbb{N}^*} |x_n|$.

Solution

1. $U_N(y) = \sum_{n=1}^{N} x_n y_n$ is a finite sum to which we can apply Hölder's inequality (P 1.5). q is associated with p by $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$|U_N(y)| \leq \left(\sum_{j=1}^{N} |x_j|^q\right)^{1/q} \left(\sum_{j=1}^{N} |y_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{N} |x_j|^q\right)^{1/q} \|y\|_p.$$

   Thus, we have $\|U_N\| \leq \rho_N$, establishing the continuity of U_N.

We choose $y = (y_n)$ such that $x_n y_n = |x_n|^q$ for $n \leq N$ and $y_n = 0$ for $n > N$. Then

$$|U_N(y)| = \sum_{n=1}^{N} |x_n|^q = \rho_N^q$$

and

$$\|y\|_p = \left(\sum_{n=1}^{N} |y_n|^p\right)^{1/p} = \left(\sum_{n=1}^{N} |x_n|^p(q-1)\right)^{1/p} = \left(\sum_{n=1}^{N} |x_n|^q\right)^{1/p} = (\rho_N)^{q/p}.$$

Thus, we obtain that

$$\rho_N^q = |U_N(y)| \leq \|U_N\| \|y\|_p = \|U_N\| (\rho_N)^{q/p},$$

which implies $\|U_N\| \geq (\rho_N)^{q-q/p} = \rho_N$, showing that $\|U_N\| = \rho_N = \left(\sum_{n=1}^{N} |x_n|^q\right)^{1/q}$.

2. Since the sequence $x = (x_n)_n$ satisfies

$$\forall y = (y_n)_n \in \ell^p, \sum_{n=0}^{+\infty} |x_n y_n| = M_y < +\infty,$$

we have

$$\forall y \in \ell^p, \forall N \in \mathbb{N}, |U_N(y)| \leq M_y,$$

and therefore

$$\sup_N |U_N(y)| \leq M_y.$$

By the Banach-Steinhaus Theorem (P 2.13), $\sup_N \|U_N\| < \infty$, which implies

$$\left(\sum_{n=1}^{+\infty} |x_n|^q\right)^{1/q} < \infty,$$

and thus $x \in \ell^q$.

3. We have

$$|(U - U_N)(y)| = \sum_{n=N+1}^{+\infty} x_n y_n$$

$$\leq \left(\sum_{n=N+1}^{+\infty} |x_n|^q\right)^{1/q} \left(\sum_{n=N+1}^{+\infty} |y_n|^p\right)^{1/p}$$

$$\leq \left(\sum_{n=N+1}^{+\infty} |x_n|^q \right)^{1/q} \|y\|_p.$$

Thus, $\|U - U_N\| \leq \left(\sum_{n=N+1}^{+\infty} |x_n|^q\right)^{1/q} \to 0$, as the remainder of a convergent series. Moreover, $\|U\| = \|(x_n)_n\|_q$.

4. Let $f \in (\ell^p)^*$ be a continuous linear form, and $(e_n)_n$ the "canonical basis" of ℓ^p. Then, for $x = (x_n)_n \in \ell^p$, we have

$$x = \sum_n x_n e_n \quad \text{and thus} \quad f(x) = \sum_n x_n f(e_n).$$

Let $f_n = f(e_n)$. The sequence $\varphi = (f_n)_n$ satisfies

$$\forall (x_n)_n \in \ell^p, \sum_n |x_n f_n| < \infty.$$

From (2), $\varphi \in \ell^q$. And from (3), $\|f\| = \|(f_n)_n\|_q = \|\varphi\|_q$.

Let $\varphi : (\ell^p)^* \to \ell^q$ be defined by $\varphi(f) = (f_n)_n = \varphi$. The map φ is linear, and we have

$$\|\varphi(f)\|_q = \|(f_n)_n\|_q = \|f\|.$$

It follows that φ is an isometry, and in particular, it is continuous and injective. For thesurjectivity, let $y = (y_n)_n \in \ell^q$. Define

$$f(x) = \sum_n x_n y_n, \quad x = (x_n)_n \in \ell^p.$$

Then f is well-defined, linear, continuous, and we have $f_n = f(e_n) = y_n$, hence $\varphi(f) = (y_n)_n = y$.

Finally, $(\ell^p)^*$ is isometrically isomorphic to ℓ^q. Thus: $1 < p < \infty \implies (\ell^p)^{**} = (\ell^q)^* = \ell^p$. We then say that ℓ^p is reflexive.

5. Let $f \in (\ell^1)^*$ and $f_n = f(e_n)$, where $(e_n)_n$ is the canonical basis of ℓ^1. For all integers $n \geq 1$, we have

$$|f_n| \leq \|f\| \|e_n\|_1 = \|f\|.$$

Thus, $(f_n)_n \in \ell^\infty$ and $\|(f_n)_n\|_\infty \leq \|f\|$.

If $x = (x_n)_n \in \ell^1$ is arbitrary, then

$$f(x) = f\left(\sum_n x_n e_n\right) = \sum_n x_n f(e_n) = \sum_n x_n f_n.$$

Therefore,

$$|f(x)| \leq \|(f_n)_n\|_\infty \|x\|_1 \quad \text{and} \quad \|f\| \leq \|(f_n)_n\|_\infty.$$

Thus, $\|f\| = \|(f_n)_n\|_\infty$.

The map $\varphi : (\ell^1)^* \to \ell^\infty$ defined by $\varphi(f) = (f_n)_n$ is well-defined, linear, and

$$\|\varphi(f)\|_\infty = \|(f_n)_n\|_\infty = \|f\|.$$

Thus, φ is an isometry, and therefore it is continuous and injective.

Now let $y = (y_n)_n \in \ell^\infty$. Define f by

$$f(x) = \sum_n x_n y_n, \quad x = (x_n)_n \in \ell^1.$$

Then f is linear and also continuous, because

$$|f(x)| \leq \sum_n |y_n||x_n| \leq \|y\|_\infty \|x\|_1.$$

Thus, $f \in (\ell^1)^*$ and $\varphi(f) = y$, which shows that φ is surjective.

Finally, we have shown that $(\ell^1)^*$ is isometrically isomorphic to ℓ^∞, written as $(\ell^1)^* = \ell^\infty$.

6. Consider the space $c_0 = \{(x_n)_n \in \ell^\infty ; x_n \to 0\}$ equipped with the norm $\|\cdot\|_\infty$, and let $f \in (c_0)^*$. Define $f_n = f(e_n)$, where $(e_n)_n$ is the "canonical basis" of c_0. Define the elements $x^{(N)}$ of c_0 by

$$x_n^{(N)} = \begin{cases} \frac{f(e_n)}{|f(e_n)|} & \text{if } n \leq N \text{ and } f(e_n) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \sum_{n=1}^N |f(e_n)| &= \sum_{n=1}^N f(e_n) \left[\frac{f(e_n)}{|f(e_n)|} \right] \\ &= f\left(\sum_{n=1}^N \frac{f(e_n)}{|f(e_n)|} e_n \right) \\ &= f(x^{(N)}) \leq \|f\| \|x^{(N)}\|_\infty = \|f\|. \end{aligned}$$

Thus, $(f_n)_n \in \ell^1$ and $\|(f_n)_n\|_1 \leq \|f\|$.

For $x = (x_n)_n \in c_0$,

$$f(x) = f\left(\sum_n x_n e_n \right) = \sum_n x_n f(e_n) = \sum_n x_n f_n.$$

Therefore,

$$|f(x)| \leq \|x\|_\infty \|(f_n)_n\|_1 \quad \text{and hence} \quad \|f\| \leq \|(f_n)_n\|_1.$$

Finally, $\|f\| = \|(f_n)_n\|_1$.

The map $\varphi : (c_0)^* \to \ell^1$, defined by $\varphi(f) = (f_n)_n$, is well-defined, linear, and

$$\|\varphi(f)\|_1 = \|(f_n)_n\|_1 = \|f\|.$$

Thus, it is an isometry, and therefore continuous and injective.

Now let $y = (y_n)_n \in \ell^1$. Define

$$f : c_0 \to \mathbb{C} \quad \text{by} \quad f(x) = \sum_n x_n y_n, \quad x = (x_n)_n \in c_0.$$

Then f is linear and continuous because

$$|f(x)| \leq \sum_n |y_n||x_n| \leq \|x\|_\infty \|y\|_1.$$

Thus, $f \in (c_0)^*$ and $\varphi(f) = y$, which shows that φ is surjective.

Finally, we have shown that $(c_0)^*$ is isometrically isomorphic to ℓ^1, written as:

$$(c_0)^* = \ell^1.$$

Remark III.3. *(a) We have $\ell^1 \subseteq c_0 \subseteq \ell^\infty$, $(\ell^1)^* = \ell^\infty$, and $(c_0)^* = \ell^1$.*

(b) $\ell^1 \subset (\ell^\infty)^*$, but $(\ell^\infty)^* \ncong \ell^1$ (See Ch. III, Ex. III.2).

(c) c_0 and ℓ^1 are separable (i.e., they contain a countable dense subset), but ℓ^∞ is not separable.

Chapter IV

Different types of convergence

Contents

IV.1 The smallest topology making continuous a collection of maps

Let X be a set, and let $(Y_i)_{i\in I}$ be a family of topological spaces. For each $i \in I$, suppose we are given a map

$$\varphi_i : X \longrightarrow Y_i.$$

Question: Is it possible to equip X with a topology that makes all the maps $(\varphi_i)_{i\in I}$ continuous?
Answer: Yes, it is possible. We can define the topology

$$\tau = \left\{ \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(W_i) \,\middle|\, W_i \text{ is open in } Y_i \right\}.$$

This topology is the coarsest (or weakest) topology on X that makes each φ_i continuous.
Given a point $x \in X$, a neighborhood basis at x in the topology τ can be constructed by taking sets of the form

$$\bigcap_{\text{finite}} \varphi_i^{-1}(V_i),$$

where V_i is a neighborhood of $\varphi_i(x)$ in Y_i.

Proposition IV.1. *[1] Let x_n be a sequence in X. Then*

$$x_n \xrightarrow[n]{} x \quad \textit{if and only if} \quad \varphi_i(x_n) \xrightarrow[n]{} \varphi_i(x) \quad \forall i \in I.$$

Proof. If $x_n \xrightarrow[n]{} x$ then $\varphi_i(x_n) \xrightarrow[n]{} \varphi_i(x)$ for all $i \in I$.
Conversely, let U be a neighborhood of x in X. According to the previous argument, there exists a neighborhood

$$x \in W = \bigcap_{i\in J \text{ (finite)}} \varphi_i^{-1}(V_i) \subset U,$$

where each V_i is a neighborhood of $\varphi_i(x)$ for all $i \in J$.
Thus, for each $i \in J$, there exists an integer N_i such that

$$\varphi_i(x_n) \in V_i \quad \text{for all } n \geq N_i.$$

Let $N = \max_{i\in J} N_i$. Then,

$$\forall n \geq N,\ x_n \in U, \quad \text{i.e.,} \quad x_n \xrightarrow[n]{} x \quad \text{for the topology } \tau.$$

□

Proposition IV.2. *[1] Let Z be a topological space, and let ψ be a map from Z to X. Then, ψ is continuous if and only if $\varphi_i \circ \psi$ is continuous from Z to Y_i for each $i \in I$.*

Proof. If ψ is continuous, then $\varphi_i \circ \psi$ is also continuous for each $i \in I$. Conversely, let U be an open set in X, and let us show that $\psi^{-1}(U)$ is open in Z. We know that U is of the form

$$U = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \varphi_i^{-1}(w_i), \quad w_i \text{ open in } Y_i.$$

Thus,

$$\psi^{-1}(U) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \psi^{-1} \circ \varphi_i^{-1}(w_i) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} (\varphi_i \circ \psi)^{-1}(w_i).$$

This is open in Z because each map $\varphi_i \circ \psi$ is continuous. $\square$

IV.2 Definition and Elementary Properties of the Weak Topology $\sigma(E, E')$

Let E be a Banach space and let $f \in E'$. We denote by $\varphi_f : E \longrightarrow \mathbb{R}$ the map defined by $\varphi_f(x) = \langle f, x \rangle$.
When f varies over E', we obtain a family $(\varphi_f)_{f \in E'}$ of maps from E to $\mathbb{R}$.

Definition IV.1. *The weak topology $\sigma(E, E')$ on E is the coarsest topology that makes all the maps $(\varphi_f)_{f \in E'}$ continuous. (In the sense of Section I with $X = E$, $\mathbb{R} = Y_i$ and $I = E'$).*

Proposition IV.3. *[1] The weak topology $\sigma(E, E')$ is Hausdorff.*

Proof. Let x_1 and $x_2 \in X$ with $x_1 \neq x_2$. We aim to construct open sets O_1 and O_2 in the weak topology $\sigma(E, E')$ such that $x_1 \in O_1$, $x_2 \in O_2$, and $O_1 \cap O_2 = \emptyset$.
According to Hahn-Banach Theorem, second geometric form, there exists $f \in E'$ and $\alpha \in \mathbb{R}$ such that

$$\langle f, x_1 \rangle < \alpha < \langle f, x_2 \rangle.$$

We define

$$O_1 = \{x \in E,\ \langle f, x \rangle < \alpha\} = f^{-1}(-\infty, \alpha)$$

and

$$O_2 = \{x \in E,\ \langle f, x\rangle > \alpha\} = f^{-1}(\alpha, +\infty)\,.$$

□

Proposition IV.4. *Let x_0, we obtain a neighborhood base of x_0 for the weak topology $\sigma(E, E')$ by considering all sets of the form*

$$V = \{x \in E,\ |\langle f_i, x - x_0\rangle| < \varepsilon,\ \forall i \in I\}\,,$$

where I is finite and $f_i \in E'$.

Proof.

$$\begin{aligned} V &= \{x \in E,\ \ |\langle f_i, x - x_0\rangle| < \varepsilon,\ \ \forall i \in I\} \\ &= \bigcap_{i \text{ finite}} \{x \in E,\ -\varepsilon + a_i < f(x_i) < \varepsilon + a_i\} \\ &= \bigcap_{i \text{ finite}} \varphi_{f_i}^{-1}(] - \varepsilon + a_i, \varepsilon + a_i[)\,. \end{aligned}$$

Thus, V is an open set for the weak topology $\sigma(E, E')$ and contains x_0. Conversely, let U be a neighborhood containing x_0 for the weak topology $\sigma(E, E')$. We know that there exists a neighborhood W of x_0, $W \subset U$, of the form $W = \bigcap_{i \text{ finite}} \varphi_{f_i}^{-1}(w_i)$, w_i is a neighborhood of $\langle f_i, x_0\rangle = a_i$ in $\mathbb{R}$. Thus, there exists $\varepsilon > 0$ such that $]a_i - \varepsilon, a_i + \varepsilon[\subset w_i$, $\forall i$ finite. Hence, $\varphi_{f_i}^{-1}(]a_i - \varepsilon, a_i + \varepsilon[) \subset \varphi_{f_i}^{-1}(w_i)$, i.e., $x_0 \in \bigcap_{i \text{ finite}} \varphi_{f_i}^{-1}(]a_i - \varepsilon, a_i + \varepsilon[) \subset W \subset U$. □

Proposition IV.5. *Let x_n be a sequence in E. We have*

(i) $x_n \underset{n}{\rightharpoonup} x$ *for* $\sigma(E, E') \iff \langle f, x_n\rangle \underset{n}{\longrightarrow} \langle f, x\rangle,\ \ \forall f \in E'$.

(ii) *If* $x_n \underset{n}{\longrightarrow} x$ *strongly, then* $x_n \underset{n}{\rightharpoonup} x$ *weakly for* $\sigma(E, E')$.

(iii) *If* $x_n \underset{n}{\rightharpoonup} x$ *for* $\sigma(E, E')$, *then* $\|x_n\|$ *is bounded and* $\|x\| \leq \liminf \|x_n\|$.

(iv) *If* $x_n \underset{n}{\rightharpoonup} x$ *for* $\sigma(E, E')$ *and* $f_n \underset{n}{\longrightarrow} f$ *strongly in* E' *(i.e.* $\|f_n - f\| \underset{n}{\longrightarrow} 0$*), then* $\langle f_n, x_n\rangle \underset{n}{\longrightarrow} \langle f, x\rangle$.

Proof. *(i)* It follows from Proposition IV.1 and the definition of the weak topology $\sigma(E, E')$.

(*ii*) It follows from (i) since

$$|\langle f, x_n\rangle - \langle f, x\rangle| \leq \|f\|\|x_n - x\|.$$

(*iii*) $\{\langle f, x_n\rangle\}$ converges for each $f \in E'$. Thus, for all $f \in E'$,

$$\{F_{x_n}(f),\ n \in \mathbb{N}\} = \{\langle f, x_n\rangle,\ n \in \mathbb{N}\} \quad \text{is bounded in } \mathbb{R}.$$

This gives, according to Banach-Steinhaus Theorem, that

$$\|F_{x_n}\| = \|x_n\| \quad \text{is bounded.}$$

Now, for $f \in E'$, we have

$$|\langle f, x_n\rangle| \leq \|f\|\|x_n\|$$

and in the limit,

$$|\langle f, x\rangle| \leq \|f\| \liminf \|x_n\|.$$

Consequently,

$$\|x\| \leq \liminf \|x_n\|.$$

□

IV.3 The *weak** topology $\sigma(E', E)$

Let E be a Banach space and E' its dual (equipped with the dual norm $\|f\| = \sup_{x\in E,\ \|x\|\leq 1} |\langle f, x\rangle|$), and let E'' be its bidual, i.e., the dual of E', equipped with the norm $\|\xi\| = \sup_{f\in E',\ \|f\|\leq 1} |\langle \xi, f\rangle|$.

On the space E', we can define two types of convergence:

(a) The strong topology (associated with the norm).

(b) The weak topology $\sigma(E', E'')$.

Now, we define a third topology on E', the $*$-topology, denoted $\sigma(E', E)$.

For each $x \in E$, consider the map $\varphi_x : E' \longrightarrow \mathbb{R}$ defined by $\varphi_x(f) = \langle f, x\rangle$.

As x varies over E, we obtain a family of maps $(\varphi_x)_{x\in E}$ from E' to $\mathbb{R}$.

Definition IV.2. *The weak topology $*$, also denoted by $\sigma(E', E)$, is the coarsest topology that makes the family of maps $(\varphi_x)_{x\in E}$ continuous.*

Since $E \subset E''$, it is clear that the topology $\sigma(E', E)$ is coarser than the topology $\sigma(E', E'')$. In other words, $\sigma(E', E)$ has fewer open (resp. closed) sets than the topology $\sigma(E', E'')$, which in turn has fewer open (resp. closed) sets than the strong topology.

Proposition IV.6. *The weak topology $*$, also denoted $\sigma(E', E)$, is separated.*

Proof. Let f_1 and f_2 be elements of E' with $f_1 \neq f_2$, so there exists an $x \in E$ such that $\langle f_1, x\rangle \neq \langle f_2, x\rangle$. We introduce α such that

$$\langle f_1, x\rangle < \alpha < \langle f_2, x\rangle$$

and define

$$O_1 = \{f \in E',\ \langle f, x\rangle < \alpha\} = \varphi_x^{-1}\left(]-\infty, \alpha[\right).$$

$$O_2 = \{f \in E',\ \langle f, x\rangle > \alpha\} = \varphi_x^{-1}\left(]\alpha, +\infty[\right).$$

O_1 and O_2 are open sets for the weak topology $\sigma(E', E)$, satisfying $f_1 \in O_1$, $f_2 \in O_2$, and $O_1 \cap O_2 = \emptyset$. □

Proposition IV.7. *We obtain a neighborhood base of $f_0 \in E'$ for the topology $\sigma(E', E)$ by considering all sets of the form*

$$V = \{f \in E',\ |\langle f - f_0, x_i\rangle| < \varepsilon,\ \forall i \in I\},$$

where I is finite, $x_i \in E$, and $\varepsilon > 0$.

Proof.

$$\begin{aligned} V &= \{f \in E',\ |\langle f - f_0, x_i\rangle| < \varepsilon,\ \forall i \in I\} \\ &= \bigcap_{i \text{ finie}} \{f \in E',\ -\varepsilon + f_0(x_i) < f_i(x) < \varepsilon + f_0(x_i)\} \\ &= \bigcap_{i \text{ finie}} \varphi_{x_i}^{-1}\left(]-\varepsilon + f_0(x_i), \varepsilon + f_0(x_i)[\right). \end{aligned}$$

Thus, V is an open set for the weak topology $\sigma(E', E)$ and contains f_0.

Conversely, let U be a neighborhood that contains f_0 for the weak topology $\sigma(E', E)$. We know that there exists a neighborhood W of f_0, $W \subset U$, of the form $W = \underset{i \text{ finie}}{\cap} \varphi_{x_i}^{-1}(w_i)$, w_i is a neighborhood of $\langle f_0, x_i \rangle = a_i$ in $\mathbb{R}$. Thus, there exists $\varepsilon > 0$ such that $]f_0(x_i) - \varepsilon, f_0(x_i) + \varepsilon[\subset w_i$, for all i finite. Therefore, $\varphi_{x_i}^{-1}(]f_0(x_i) - \varepsilon, f_0(x_i) + \varepsilon[) \subset \varphi_{x_i}^{-1}(w_i)$, i.e.

$$f_0 \in \underset{i \text{ finie}}{\cap} \varphi_{x_i}^{-1}(]f_0(x_i) - \varepsilon, f_0(x_i) + \varepsilon[) \subset W \subset U.$$

$\square$

Notation: Given a sequence $(f_n)_n$ in E', we denote by $f_n \overset{*}{\rightharpoonup} f$ the convergence of f_n to f for the topology $\sigma(E', E)$.
To avoid confusion, we will often specify $f_n \overset{*}{\rightharpoonup} f$ for $\sigma(E', E)$, $f_n \rightharpoonup f$ for $\sigma(E'', E')$, and $f_n \longrightarrow f$ strongly.

Proposition IV.8. *Let $(f_n)_n$ be a sequence in E'. We have the following*

(i) $f_n \overset{*}{\rightharpoonup} f$ *for* $\sigma(E', E) \iff \langle f_n, x \rangle \underset{n}{\longrightarrow} \langle f, x \rangle$, $\forall x \in E$.

(ii) If $f_n \underset{n}{\longrightarrow} f$ *strongly, then* $f_n \rightharpoonup f$ *weakly for* $\sigma(E', E'')$. *If* $f_n \rightharpoonup f$ *weakly for* $\sigma(E', E'')$, *then* $f_n \overset{*}{\rightharpoonup} f$ *for* $\sigma(E', E)$.

(iii) If $f_n \overset{*}{\rightharpoonup} f$ *for* $\sigma(E', E)$, *then* $\|f_n\|$ *is bounded and* $\|f\| \leq \liminf \|f_n\|$.

(iv) If $f_n \overset{*}{\rightharpoonup} f$ *for* $\sigma(E', E)$ *and if* $x_n \underset{n}{\longrightarrow} x$ *strongly in* E *(i.e.* $\|x_n - x\| \underset{n}{\longrightarrow} 0$*), then* $\langle f_n, x_n \rangle \underset{n}{\longrightarrow} \langle f, x \rangle$.

Proof. (i) It follows from Proposition IV.1 and the definition of the *weak** topology $\sigma(E', E)$.

(ii) $f_n \rightharpoonup f$ since $|\langle F, f_n \rangle - \langle F, f \rangle| \leq \|F\| \|f_n - f\|$ for all $F \in E''$. in particular, $f_n(x) \to f(x)$ for all $x \in E$ which means that $f_n \overset{*}{\rightharpoonup} f$ for $\sigma(E', E)$.

(iii) $\{\langle f_n, x \rangle\}$ converges for each $x \in E$. Thus, for all $f \in E'$,

$$\{\langle f_n, x \rangle,\ n \in \mathbb{N}\} \quad \text{is bounded in } \mathbb{R}.$$

This gives, according to Banach-Steinhaus Theorem, that

$$\|f_n\| \quad \text{is bounded.}$$

Now, for $x \in E$, we have

$$|\langle f_n, x \rangle| \leq \|f_n\| \|x\|$$

and in the limit,

$$|\langle f, x\rangle| \leq \|x\| \liminf \|f_n\|.$$

Consequently,

$$\|f\| \leq \liminf \|f_n\|.$$

(iV) By the triangle inequality,

$$\begin{aligned}|f_n(x_n) - f_n(x)| &\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \|f_n\| \, \|x_n - x\| + |f_n(x) - f(x)| \\ &\to 0\end{aligned}$$

□

IV.4 Exercises

Exercise 01: Let E be a normed space and let $C \subseteq E$ be a convex subset.

(a) Show that C is weakly closed if and only if it is strongly closed.

(b) Show that the weak closure of C is equal to the strong closure of C.

Exercise 02: Let E and F be two normed spaces, T a continuous linear operator from E into F, and (x_n) a sequence in E.

(a) Show that if $x_n \rightharpoonup x$ (weak convergence), then $T(x_n) \rightharpoonup T(x)$ (weak convergence).

(b) Furthermore, if T is compact, then $T(x_n) \to T(x)$ (strong convergence).

Exercise 03: Let E and F be two Banach spaces and let $T : E \to F$ be a linear operator.

(a) Show that the following conditions are equivalent:

  (i) $T : (E, \|\cdot\|) \to (F, \|\cdot\|)$ is continuous (T is continuous strong-strong).

  (ii) $T : (E, \sigma(E, E^*)) \to (F, \sigma(F, F^*))$ is continuous (T is continuous weak-weak).

  (iii) $T : (E, \|\cdot\|) \to (F, \sigma(F, F^*))$ is continuous (T is continuous strong-weak).

(b) Show that the following two conditions are equivalent:

(iv) $T : (E, \sigma(E, E^*)) \to (F, \|\cdot\|)$ is continuous (T is continuous weak-strong).

(v) T is continuous strong-strong and $T(E)$ is finite-dimensional (T is a finite rank bounded linear operator).

IV.5 Solved problem

Let $E = C([0,1])$ be the space of continuous real-valued functions on the interval $[0,1]$, equipped with the supremum norm $\|u\|_\infty = \sup_{0\leq t\leq 1} |u(t)|$.

(a) Define, for each $t \in [0,1]$, the functional $\epsilon_t(u) = u(t)$. Show that $\epsilon_t \in E^*$, where E^* is the dual space of E, and compute the norm $\|\epsilon_t\|$.

(b) Show that if a sequence (u_n) converges weakly to u in E, then it also converges pointwise to u.

(c) Demonstrate that the converse does not hold by considering the sequence $u_n(t) = n^2 t(1-t)^n$. Specifically, show that this sequence converges pointwise but not weakly.

(d) Analyze the pointwise convergence of the sequence (v_n) defined by

$$v_n(t) = \begin{cases} -nt + 1 & \text{if } 0 \leq t \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < t \leq 1. \end{cases}$$

Show the pointwise behavior of v_n.

(e) Deduce from the previous results that the space $C([0,1])$ is not reflexive.

(f) Using similar reasoning, deduce that $L^\infty([0,1])$ is also not reflexive.

Solution

(a) It is clear that ϵ_t is linear. Next, observe that

$$|\epsilon_t(u)| \leq \|u\|_\infty.$$

It follows that ϵ_t is bounded and $\|\epsilon_t\| \leq 1$. Now, if v is the constant function equal to 1, then $\|v\|_\infty = 1$ and $\epsilon_t(v) = 1$. Therefore,

$$\|\epsilon_t\| = \sup_{\|u\|_\infty = 1} |\epsilon_t(u)| \geq |\epsilon_t(v)| = 1.$$

Thus, $\|\epsilon_t\| = 1$.

(b) Let $t \in [0,1]$. If (u_n) converges weakly to u, then $f(u_n) \to f(u)$ for all $f \in E^*$. In particular,

$$\epsilon_t(u_n) \to \epsilon_t(u).$$

This implies that $u_n(t) \to u(t)$ in $\mathbb{R}$. Since t was arbitrary, this shows that (u_n) converges pointwise to u.

(c) A study of the variation of u_n shows that

$$\|u_n\|_\infty = u_n\left(\frac{1}{n+1}\right) = \frac{n^2}{n+1}\left(1 - \frac{1}{n+1}\right)^n.$$

Therefore, (u_n) is unbounded and cannot converge weakly.

(d) Draw the graph of v_n for some n. Let $t \in [0,1]$. If $t = 0$, then $v_n(0) = 1$, so $v_n(0) \to 1$. If $t > 0$, choose $n_0 > \frac{1}{t}$. For $n \geq n_0$, we have $t > \frac{1}{n}$, so $v_n(t) = 0$. Therefore, $v_n(t) \to 0$ as $n \to \infty$. It follows that (v_n) converges pointwise to the discontinuous function v defined by

$$v(t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } 0 < t \leq 1. \end{cases}$$

(e) Suppose that $C[0,1]$ is reflexive. Note that $\|v_n\| = 1$, so (v_n) is bounded. Eberlein –Smulyan Theorem, there exists a subsequence (v_{n_k}) that converges weakly to some $w \in C[0,1]$. By (b), (v_{n_k}) converges pointwise to w. By (d), (v_{n_k}) converges pointwise to v. Therefore, $w = v$. This is a contradiction because w is continuous and v is not.

(f) We know that $C[0,1]$ is complete. Therefore, it is closed in $L^\infty([0,1])$. If $L^\infty([0,1])$ were reflexive, then $C[0,1]$ would also be reflexive, which is a contradiction.

Bibliography

[1] H. BREZIS. *Analyse fonctionnelle. Théorie et application.* Masson, 1983.

[2] J. Conway. *A Course in Functional Analysis.* Springer, 1990.

[3] H. Gebran. *Functional Analysis.* www.hichamgebran.com, 2022.

[4] M. Hazi. *Visite guidé dans les espaces normés.* Office des publications universitaires, 2015.

[5] M. Fabian et al. *Banach space theory.* Springer, 2011.

[6] D. Li. *Cours d'analyse fontionnelle.* Ellipses, 2013.

[7] D. Manceau, S. Maingot. Théorie spectrale.

[8] R. Meise and D. Vogt. Introduction to functional analysis. *Oxford Science publications*, , 1997.

[9] E. S. Suhubi. *Fonctional Analysis.* Kluwer Academic, 2003.

[10] V. Kadets. *A Course in Functional Analysis and Measure Theory.* 2018.

[11] A. Kolmogorov, S. Fomine. *Éléments de la théorie des fonctions et de l'analyse fonctionnelle.*

[12] E. Kreyszig. *Introductory functional analysis with applications.* John Willey, 1978.

[13] W. Rudin. *Functional Analysis.* McGraw-Hill, 1991.

[14] H. L. Royden, P.M. Fitzpatrick. *Real Analysis.* Pearson, 2017.

[15] K. Yosida. *Functional Analysis.* Springer, 1995.